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GOLDEN PROPORTION IN THE STRING QUARTETS OF BELA BARTOK

by

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ABSTRACT

The purpose of this study is to discover the extent to which golden proportion is evident in the form of each movement of Bartok's six String Quartets, and to elucidate the relationship between this formal principle and traditional music forms as used by Bartok. The findings indicate that Bartok used golden proportion in every quartet, in unexpected and innovative ways, and that such use was intentional.

Only eight of the twenty-two quartet movements (considering the Third Quartet as one large "movement") exhibit golden section of their entire length; in only one case (the extremely brief Introduzione to the third movement of the First Quartet) are the resulting sections further subdivided to form a modular hierarchy. More typically, Bartok divides a formal structure into blocks and applies the principle of golden section to proportion each one individually. For example, a sonata form may be divided into four blocks, the exposition being the first, the development subdivided to form two more, and the recapitulation and coda together constituting the fourth block; each block will be in golden proportion internally but will not be in golden proportion to the other blocks or to the length of the whole movement. If the whole movement is also in golden proportion, it will divide at either the point of golden section or the end point of one of the blocks. Thus, a two-level scheme of aligned but proportionally unrelated golden systems results.

In most cases, at least one of the blocks will be subdivided in a manner other than simple golden section into two modules. Contained, interlocking, overlapping, linearly-extended, and diagonally-related systems are described and discussed as more complex methods of golden proportioning invented by Bartok.

It is argued that Bartok used golden proportion as an aid to composition, based in part on the evidence of proportionally unrelated sections which appear to have been later additions to an original fully proportioned scheme. However, there is also evidence that Bartok used golden proportion to reinforce musical relationships in the block architecture of his works.

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CHAPTER I: INTRODUCTION

PROPORTION, GOLDEN SECTION, AND THE FIBONACCI SERIES

> Proportion 1. A portion or part in its relation to the whole; a comparative part, a share; sometimes simply, a portion, division, part.
>
> 2. The relation existing between things or magnitudes as to size, quantity, number or the like; comparative relation, ratio.[1]

Proportion has surely been an issue of concern to everyone at various times, from choosing the largest of the available pieces of cake as a child—touching them all in the process—to politely selecting the smallest piece as an adult; from finding a house large enough for a growing family to deciding if a smaller house would be more appropriate now that the children have left home—but with enough room for the grandchildren to visit!

Proportion is a more conscious concern while hanging pictures on a wall, searching for that harmonious impression to be had only when everything is in balance. This was no doubt the concern of the artist too, as he sought an arrangement of the picture's subject matter so that the size, placement, direction and color of each part would contribute to a satisfying whole.

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[1] Oxford English Dictionary (Compact Edition, 1971), p. 2329.

Proportion was of concern to the ancient Greeks as well, and they predictably based their approach to it on the mathematics of geometry. From ancient times it was believed that there was special aesthetic appeal in the ratio between the sides of a square and its diagonal—$1:\sqrt{2}$ or 0.707... (Figure 1).

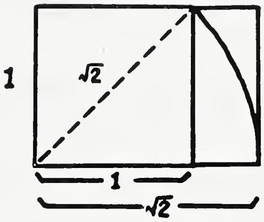


Figure 1. The ratio $1:\sqrt{2}$

Plato made a similar claim for the relationship between the sides of a rectangle formed by the two halves of an equilateral triangle—$1:\sqrt{3}$ or 0.577... (Figure 2).

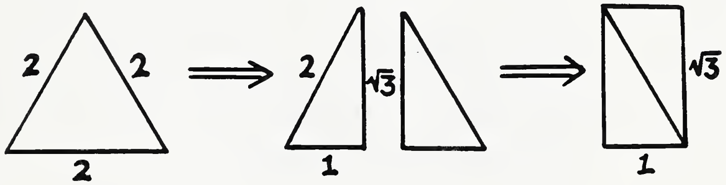
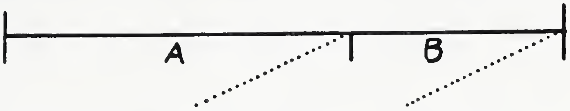
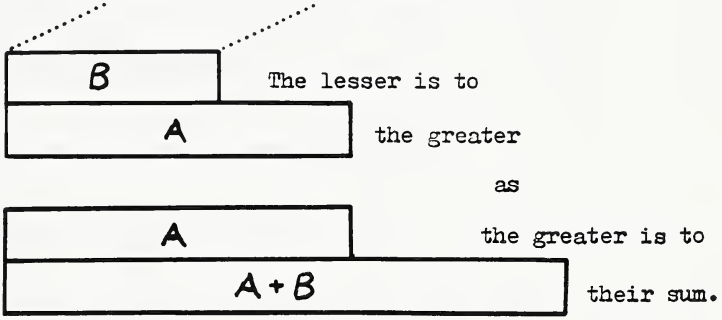


Figure 2. The ratio $1:\sqrt{3}$

One geometric relationship of ancient origin was believed to be of such significance that it was called "divine proportion." When a quantity is divided so that "the lesser is to the greater as the greater is to their sum" (Figure 3), it is said to be divided by sectio aurea—golden section.



a. Line segment divided by golden section



$$B : A :: A : A + B$$

or

$$\frac{B}{A} = \frac{A}{A + B}$$

b. Comparison of proportions

Figure 3. Golden section

Italian mathematician Luca Pacioli's De Divina Proportione ("On Divine Proportion") of 1509, illustrated by Leonardo da Vinci, stressed the aesthetic value of this relationship, and derived the equation for the golden number (the ratio of the whole quantity to that of the greater part), designated "Ø" (phi; Figure 4).

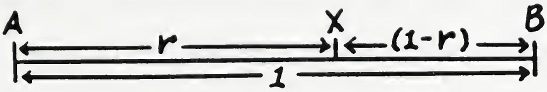


Figure 4. Derivation of ϕ

When the length of a line segment $\overline{AB}$ is 1, and the longer portion of it, $\overline{AX}$, is of length *r*, then the ratio of *r* to (1 - *r*) is the same as the ratio of 1 to *r*, or:

$$\phi = \frac{1}{r} = \frac{r}{1 - r}$$

$$r^2 = (1 - r)$$

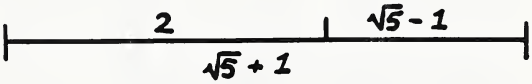
$$r^2 + r - 1 = 0$$

$$(r + \frac{\sqrt{5} + 1}{2})(r - \frac{\sqrt{5} - 1}{2}) = 0$$

$$r = \frac{\sqrt{5} - 1}{2} \quad (\text{or} \; -\frac{\sqrt{5} + 1}{2}, \text{ a negative value})$$

$$\phi = \frac{1}{r} = \frac{2}{\sqrt{5} - 1} = \frac{\sqrt{5} + 1}{2} = 1.618\ldots$$

Thus ϕ is given by the ratio $(\sqrt{5} + 1):2$, which equals 1.618...;[2] its reciprocal, $(\sqrt{5} - 1):2$, gives the reverse ratio—that of the smaller part to the larger part—and is equal to 0.618... (Figure 5).



$$\phi = (\sqrt{5} + 1):2 = 2:(\sqrt{5} - 1) = 1.618\ldots$$

$$1/\phi = (\sqrt{5} - 1):2 = 2:(\sqrt{5} + 1) = 0.618\ldots$$

Figure 5. ϕ and its reciprocal ($1/\phi$)

---

[2] ϕ is an irrational number approximated by 1.618033988750 to twelve decimals. The three-decimal approximation (1.618) varies by only 0.0021%, but the six-decimal form (1.618034), correct to within $7 \times 10^{-7}\%$, is preferred when powers of ϕ are to be computed.

A geometrical figure exhibiting golden ratios may easily be constructed, based on a square with sides of two units in length (Figure 6). The center point of one side is found, thus dividing that side into two segments of one unit each. A diagonal is now drawn between this center point and one of the opposing corners, effectively creating a right triangle sharing sides of one and two units with the square, and having a hypotenuse of $\sqrt{5}$ units. If the hypotenuse is used as the radius of an arc and the divided side of the square is produced to this arc, a line segment of $\sqrt{5} + 1$ units will have been constructed, divided by golden section into segments of 2 and $\sqrt{5} - 1$ units. Extending the opposite side of the square similarly creates a golden rectangle, $\sqrt{5} + 1$ by 2 units. If the original square is removed, what remains is another golden rectangle, 2 by $\sqrt{5} - 1$ units.

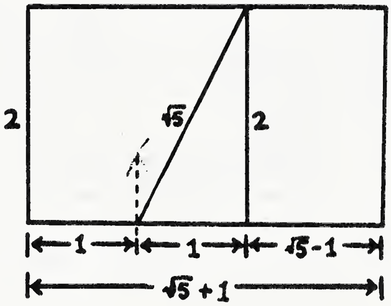
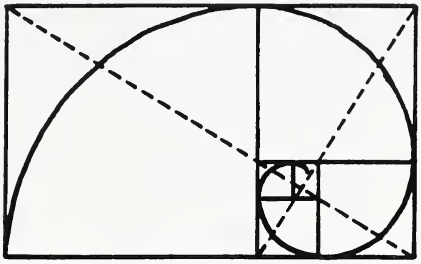
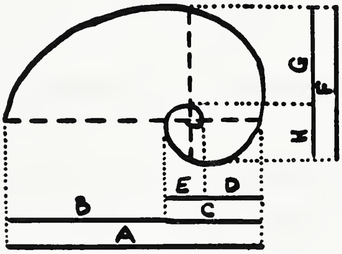


Figure 6. Constructing a golden rectangle

If this process is continued and circular arcs are drawn in each square (Figure 7a), the resulting curve approximates a logarithmic spiral, which intersects all its radiants at the same angle and at points of golden section (Figure 7b). The growth of seashells and the horns of some ruminants follows a logarithmic spiral.



a. Approximation with golden rectangles



A:B :: B:C :: C:D :: D:E :: F:G :: G:H :: Ø:1

or

$$\frac{A}{B} = \frac{B}{C} = \frac{C}{D} = \frac{D}{E} = \frac{F}{G} = \frac{G}{H} = \varnothing$$

b. Golden section in the logarithmic spiral

Figure 7. The logarithmic spiral

Golden proportion may be found in several other geometrical constructions as well. Euclid showed that the ratio between the radius of a circle and a side of the regular decagon it circumscribes is ϕ (Elements, vi, 30). More simply, the ratio between the diagonals of a regular pentagon and its sides is ϕ (Figure 8a); the intersection of two diagonals divides each of them by golden section as well (Figure 8b); furthermore, a third intersecting diagonal subdivides one of the larger sections in golden proportion (Figure 8c).

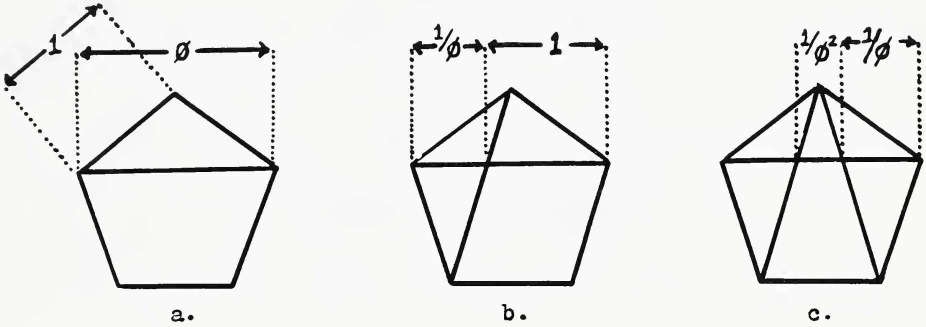


Figure 8. Diagonals of a regular pentagon

In fact, the intersection of all the diagonals of a regular pentagon forms a pentagram, a figure which exhibits golden section in all its proportions (Figure 9). It may have been due to this fact that belief in the "divine" nature of golden proportion arose, since the pentagram is an age-old mystic symbol credited with magical virtues. Among the powers attributed to this five-pointed star was the repulsion of satanic forces, an idea referred to in Goethe's Faust (Part I, Act 1, scene i), presumably because "the 'five straight lines' of which the figure consists form one continuous line or 'endless knot.' "[3]

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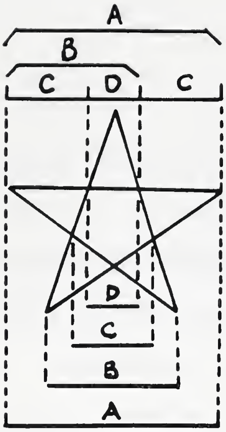
[3] Oxford English Dictionary, p. 2123.



$$A:B :: B:C :: C:D :: \phi:1$$

or

$$\frac{A}{B} = \frac{B}{C} = \frac{C}{D} = \phi$$

Figure 9. The pentagram

Symmetry 1. Mutual relation of the parts of something in respect of magnitude and position. 4

The pentagram, the diagonals of a regular pentagon, and even the golden rectangle illustrate another aspect of golden section: the process is hierarchical. Golden subdivision can be continued ad infinitum. If a line segment is divided into a larger and a smaller section in golden proportion (Figure 10a), subdividing the larger section by golden section yields a subsection the same size as the smaller section (Figure 10b); subdividing the smaller section (or further subdividing the larger subsection) similarly yields a subsection equal in size to the smaller subsection of the previous operation (Figure 10c), and so on (Figure 10d).

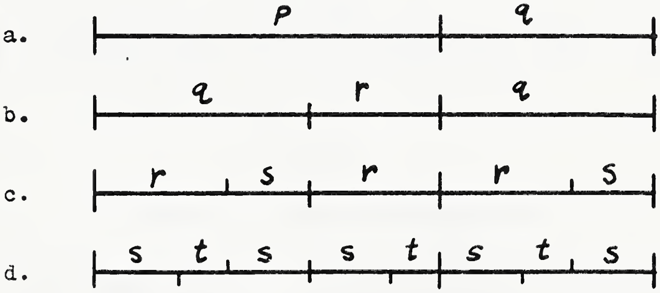


Figure 10. Continuous subdivision

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4 Oxford English Dictionary, p. 3207.

Furthermore, the process is symmetrical. Different orders of subdivision (large/small or small/large) at each level can lead to the same result (Figure 11, a and b; cf. Figure 10). In specific, a line segment divided by golden section displays reflexive congruence: reflection in an axis carries a "larger-section-first" segment into a "smaller-segment-first" segment, or a left-handed segment into a right-handed segment (Figure 12).

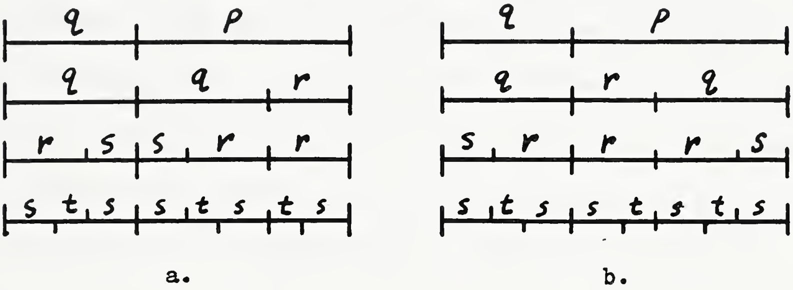


Figure 11. Differing orders of subdivision

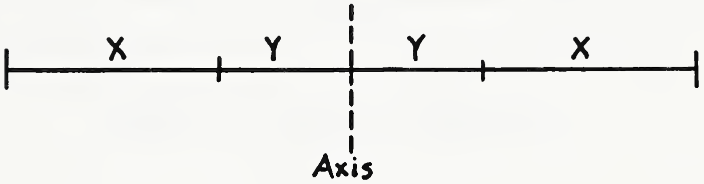
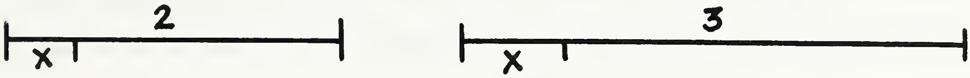


Figure 12. Reflexive congruence

Due to this "handed-ness" of golden section, the order of the larger and smaller sections is significant. A line segment divided such that the larger section is first might be termed "positive"; its reflection, having the larger section last, would then be referred to as "negative." These terms suggest the scientific concept of polarity—magnetic or electrical forces equal in strength but opposite

in direction; conveniently, they may be abbreviated by plus and minus signs.[5]

Description of the various levels of golden subdivision becomes difficult and confusing without a conventional terminology. One method of dealing with the hierarchies of golden section is to speak of "modules," a term borrowed from the classic orders of architecture, where it signifies "the unit of length by which the proportions of the parts are expressed."[6] In the terminology of golden section, "module" might be defined as the unit of proportion by which the length of the parts are expressed, i.e., a certain portion of the whole. For example, if module x is defined as one-fifth of the length of a line segment, then the length of the module increases as the length of the whole segment increases, but its proportion to the whole is constant: one-fifth (Figure 13).



a. x = 0.4 (1/5 of 2)   b. x = 0.6 (1/5 of 3)

Figure 13. A module of proportion

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[5] Other writers have chosen various alternate word pairs, such as "active/passive" (Pascoe) or "major/minor" (Hambidge), but these may tend to arouse emotive connotations. Clive B. Pascoe, "Golden Proportion in Musical Design" (D.M.E. dissertation, University of Cincinnati, 1973), argues against the use of "positive" and "negative" on the grounds that "there cannot be 'negative' quantity in functional design" (p. 55). However, he extends the use of word pairs from the description of the type of division—larger portion first or last—to the naming of the sections themselves—the larger portion as "active," the smaller, "passive"—hence his rejection of a "negative" section. The present study restricts the use of "positive" and "negative" to the former context, wherein the concept of polarity is appropriate; if another word pair is still to be preferred, perhaps "anterior" and "posterior" would serve.

[6] Oxford English Dictionary, p. 1829.

Since division by golden section yields two unequal parts—a greater and a lesser—the description of any quantity will be in terms of two different modules. Similarly, the subdivision of a module may be described with reference to two smaller modules. For example, in Figures 10 and 11, each line segment may be described as being divided into modules *p* and *q* (positively in Figure 10, negatively in Figure 11); similarly, module *p* is divided into modules *q* and *r* (positively in Figures 10 and 11a, negatively in Figure 11b). At the fourth level of subdivision, each segment is divided into five *s* modules and three *t* modules.

The proportion of the whole which any module represents is easily determined, since each module is a golden section of the previous (next larger) one. Thus the first module is $1/\phi$ or 0.618 of the whole; the second module is $1/\phi$ of the first module, which is $1/\phi^2$ or 0.382; the third module is $1/\phi$ of the second module, which is $1/\phi^3$ or 0.236; and so on (Figure 14).[7] If each module is assigned a number (rather than a letter as in Figures 10 and 11), a general formula may be stated which determines a module's ratio to the whole from its number: module $n = 1/\phi^n$. (The letters used in Figures 10 and 11 are also indicated in Figure 14 for ease of comparison.)[8]

---

[7] All figures are given to three significant digits, but computation of the inverse powers of ϕ were made to six digits for greater accuracy.

[8] This system of numbering, which relates a module's number to the inverse power of ϕ which defines it and which recognizes that the process of subdivision may continue to infinity, differs from that used by Pascoe, who arbitrarily divided a length into fourteen modules (the eight given here plus their reflections) but allowed for the smaller modules at the extremes only—no subdivision of the central module, for example, is provided for.

Module		Proportion	
Letter	Number	Formula	Decimal
p	1	$1/\phi$	0.618
q	2	$1/\phi^2$	0.382
r	3	$1/\phi^3$	0.236
s	4	$1/\phi^4$	0.146
t	5	$1/\phi^5$	0.0902
	6	$1/\phi^6$	0.0557
	7	$1/\phi^7$	0.0344
	8	$1/\phi^8$	0.0213
	.	.	.
	.	.	.
	.	.	.
	n	$1/\phi^n$	

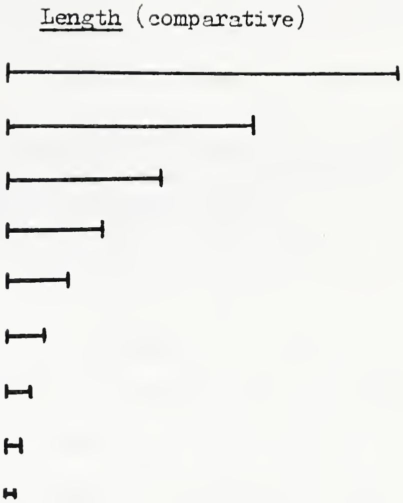

Figure 14. Golden modules

> Series 7. Math. A set of terms in succession (finite or infinite in number) the value of each of which is determined by its ordinal position according to a definite rule known as the law of the series; esp. a set of such terms continuously added together. 9

This system of modules actually forms a series: the value of each module (its proportion to the whole) is determined by its number (ordinal position) according to the law "module n = $1/Øn$," which can be calculated for an infinite number of terms. (The series is said to be "convergent," since its terms approach zero as n approaches infinity.) In fact, it forms a geometric series: the ratio between any term and the preceding one is constant $(1/Ø)$.[10]

Naturally, if the reciprocal of each term is found, another geometric series is formed; this is the Ø series, governed by the law "term n $(U_n) = Ø^n$," with the ratio between any two terms constant at Ø. In other words, each term is in golden proportion to the terms on either side of it. Table I shows this series calculated in two ways: first using the exact formula for $Ø = (\sqrt{5} + 1)/2$, then using its decimal approximation.

Each term of a geometric series may be multiplied or divided by any constant without affecting the ratio between the terms; an example is included in Table I—the Ø series divided by $\sqrt{5}$ $(U_n = Ø^n/\sqrt{5})$.

---

9 Oxford English Dictionary, p. 2737. Some authorities prefer the terms "sequence" or "progression," reserving "series" for the latter usage.

[10] Two other types of series are the arithmetic, in which the difference between any two consecutive terms is constant, and the harmonic, in which the terms' reciprocals form an arithmetic series.

TABLE I

The Ø Series and Related Series

Term Number	1	2	3	4	5	6	. . . n
Ø Series	$\frac{\sqrt{5}+1}{2}$	$\frac{\sqrt{5}+3}{2}$	$\frac{2\sqrt{5}+4}{2}$	$\frac{3\sqrt{5}+7}{2}$	$\frac{5\sqrt{5}+11}{2}$	$\frac{8\sqrt{5}+18}{2}$	. . . Ø^n
	1.618	2.618	4.236	6.854	11.09	17.94	
Ø Series$/\sqrt{5}$	0.7236	1.171	1.894	3.065	4.960	8.025	. . . Ø$^n/\sqrt{5}$

The first or last term of a series is called an *extreme*;[11] of three quantities forming a geometric series, the middle one is called the *mean proportional*.[12] Since any quantity divided by golden section forms a three-term geometric series with its larger and smaller parts, golden proportion is sometimes referred to as *extreme and mean ratio*, and the middle term (or the point on a line determined by it) is called the *golden mean*.[13]

Another type of series is the *recursive series*, "of which any term is formed by the addition of a certain number of preceding terms, multiplied or divided by any determinate numbers whether positive or negative."[14] In order uniquely to define a recursive series, the law governing the series and a number of adjacent terms must be specified.

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[11] Oxford English Dictionary, p. 941.

[12] Ibid., p. 1753.

[13] Not to be confused with the Greek principle of the golden mean, "the avoidance of excess in either direction (tr. L. *aurea mediocritas*, Hor. *Odes* II.x.5)" (ibid., p. 1171, s.v. "Golden," Sc.).

[14] Ibid., p. 2449, s.v. "Recurring," citing Encyclopedia Britannica (ed. 3) XVII.297/1. Variously called "recurring," "recurrent," or "summation" series.

The simplest type of recursive series is that in which each term is the sum of the preceding two terms; that is,

$$U_n = U_{n-1} + U_{n-2}.$$

For example, given the first two terms

$$U_1 = 1$$

$$U_2 = 1,$$

the following series is derived:

(1) 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, . . .

If, instead, the first two terms are

$$U_1 = 1$$

$$U_2 = 3,$$

a different series results:

(2) 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, . . .

The series at (1) was first described by Italian mathematician Leonardo of Pisa (born Fibonacci, an abbreviation of _filius Bonacci_, "son of Bonaccio"). His work _Liber Abaci_ ("Book of the Abacus") of 1202 (second version, 1228; published, 1857) was a compilation of the mathematical knowledge he gained while in North Africa, and was responsible for the introduction of Arabic numerals to Europe. One of the many sample problems was the calculation of the number of pairs of rabbits born of one pair in a year (assuming a pair produces another pair every month, beginning two months after their birth); the first series above indicates the number of pairs born each month, so the sum of the first twelve terms gives the number born in a year (376).

Édouard Lucas, who did much work on recursive sequences in the late nineteenth century, first named this series after Fibonacci; the series at (2) above was identified and discussed by Lucas, and was named after him.[15] Its significance may be seen by comparing it with the ∅ series from Table I: the Lucas series constitutes the closest whole number approximation of the ∅ series (except for the first term, which is adjusted to form a recursive series; Table II).

Furthermore, the Fibonacci series is the closest whole number approximation of the ∅ series divided by $\sqrt{5}$ (Table III).

In each case, the whole number approximation is alternately higher and lower than the actual geometric series value, and becomes more exact with each term.[16]

Since the ratio between consecutive terms of both geometric series (the ∅ series and the ∅ series divided by $\sqrt{5}$) is constant at ∅, the term-to-term ratio of their approximations (the Lucas and Fibonacci series, respectively) also approaches ∅; that is, each term is the golden mean of the next term, rounded to the nearest whole number.[17] Table IV shows that for both recursive series the ratio between each term and its predecessor approximates ∅ to three decimal places by the tenth term.

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[15] One of the many interrelationships between these two series may be noted in the first statement of the ∅ series in Table I: the successive coefficients of the radical $\sqrt{5}$ are from the Fibonacci series while the second term of each numerator is from the Lucas series.

[16] See Appendix I for a precise statement of the divergence.

[17] In fact, any recursive series of the form

$$U_n = U_{n-1} + U_{n-2}$$

will approximate a series of golden means in its higher terms; the closer the ratio between the first two terms is to ∅, the better will be the approximation of the related geometric series.

TABLE II

The Ø Series and the Lucas Series

Ø Series	1.618	2.618	4.236	6.854	11.09	17.94	. . .
Lucas Series	1	3	4	7	11	18	. . .

TABLE III

The Ø Series/√5 and the Fibonacci Series

Ø Series/√5	0.7236	1.171	1.894	3.065	4.960	8.025	. . .
Fibonacci Series	1	1	2	3	5	8	. . .

TABLE IV

Term-to-term Ratios in the Lucas and Fibonacci Series

	Term Number	1	2	3	4	5	6	7	8	9	10
Lucas Series	Term	1	3	4	7	11	18	29	47	76	123
	Ratio		3	1.333	1.750	1.571	1.636	1.611	1.621	1.617	1.618
Fibonacci Series	Term	1	1	2	3	5	8	13	21	34	55
	Ratio		1	2	1.5	1.667	1.600	1.625	1.615	1.619	1.618

In fact, the Fibonacci series is related to a definition of the golden mean: 1/Ø can be determined as

> the limit of the continued fraction $\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\dots}}}$
>
> of which the partial fractions 1/1, 1/2, 2/3, 3/5, 5/8, 8/13, ... are approximations, alternately larger and smaller than [1/Ø], each a better approximation than its predecessor. 18

The Fibonacci series is also related to a number of natural phenomena (other than the propagation of rabbits). Botanists have found that the most common type of leaf arrangement (or "phyllotaxis") is the spiral, in which there is one leaf at each node, the nodes forming a spiral about the stem; this pattern is often expressed by a fraction (or "fractional divergence") indicating the portion of a full revolution between successive leaves. For example, the phyllotaxis of corn is 1/2; of alder, hazel, and beech, 1/3; of oak, cherry, apple, and Pointsettia, 2/5; of rose, raspberry, and barberry, 3/8; of various other plants, 5/13 and 8/21.[19] These fractions fall into a series of which the numerators and denominators both form part of the Fibonacci series. The arrangement of the whorls on a pinecone, fruitlets on a pineapple, florets on a sunflower, and petals on a camomile and even the lowly dandelion have been claimed to follow a sequence of Fibonacci numbers. Johannes Kepler's study of phyllotaxis prompted him to call the Fibonacci series (or the principle of golden proportion which it represents) "an idea used by the Creator to generate the similar from the similar" (1611).[20]

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18 Dirk J. Struik, "Golden Section," Encyclopedia Americana (Canadian edition), XIII, 28.

19 Encyclopedia Americana, XVII, 170-71, s.v. "Phyllotaxis."

20 Ibid.

Æsthetic(s) 1. The science which treats of the conditions of sensuous perception. *Obs.*
2. The philosophy or theory of taste, or of the perception of the beautiful in nature and art. 21

Golden proportion has also been used by human "creators" throughout history, perhaps beginning with the builders of Egypt's pyramids. "A certain obscure passage in Herodotus [implies] that the area of each triangular face of the [Great] Pyramid is equal to the square of the vertical height; and this accords well with the actual facts. If this is so, the ratios of height, slope and base can be expressed in terms of the 'golden section.' "[22] Assuming the base of such a pyramid to be 2 units, the height of each face would be Ø, while the vertical height of the pyramid and the slope of each face would be √Ø (Figure 15). Further, the total area of the faces would be Ø times the area of the base.

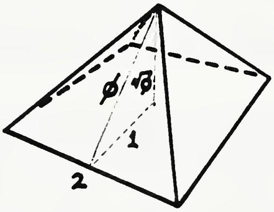

Figure 15. A golden pyramid

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21 Oxford English Dictionary, p. 37.

[22] Herbert Westren Turnbull, "The Great Mathematicians," in *The World of Mathematics*, ed. James R. Newman (New York: Simon & Schuster, 1956), I, 80.

Artists throughout the ages have tried to determine the "correct" proportions of the human body.[23] Just as an architect might measure Greek ruins or pore over Vitruvius' codification of the various "orders"[24] in an attempt to achieve the same harmony of design, painters and sculptors would study classical sculpture, such as Polyclitus' Doryphorus ("Spear Bearer," known as "the Canon" because it embodied the proportions of the ideal male figure), in an attempt to recapture the elusive quality of beauty that distinguished these gods created by human hands.

Fra Luca Pacioli's treatise De Divina Proportione of 1509 (for which his friend Leonardo da Vinci provided diagrams of the five regular Euclidean polyhedra) applied the principle of golden section to human proportion, and included measurements of classical statues for comparison with the Vitruvian canon. The first German theorist of human proportions, Albrecht Dürer, experimented with at least twenty-six different sets of proportions;[25] in his engraving The Fall of Man (or Adam and Eve), Dürer depicts the first two beings created in the image of God as embodying "divine" proportion (Figure 16).[26]

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[23] For an excellent summary see Erwin Panofsky, "The History of the Theory of Human Proportions as a Reflection of the History of Styles," in his Meaning in the Visual Arts (Garden City, N.Y.: Doubleday, 1955), pp. 55-107.

[24] Vitruvius, De Architectura, trans. as "The Ten Books on Architecture" by Morris Hicky Morgan (New York: Dover, 1914, rpt. 1960); see especially Book III, chapter I (pp. 72-75), "On Symmetry: In Temples and In the Human Body."

[25] Albrecht Dürer, Vier Bücher von menschlicher Proportion (Nuremberg, 1528), 4 vols.

[26] Dürer may not consciously have been using golden section here, but it is clear that he was seeking the ideal male and female figures based on drawings of the Apollo Belvedere and the Venus de Milo (Aphrodite from Melos): the preliminary drawing W. 333 (one of eight) presents the figures clearly against a black background; the engraving was carefully done in stages, as several trial impressions are extant;

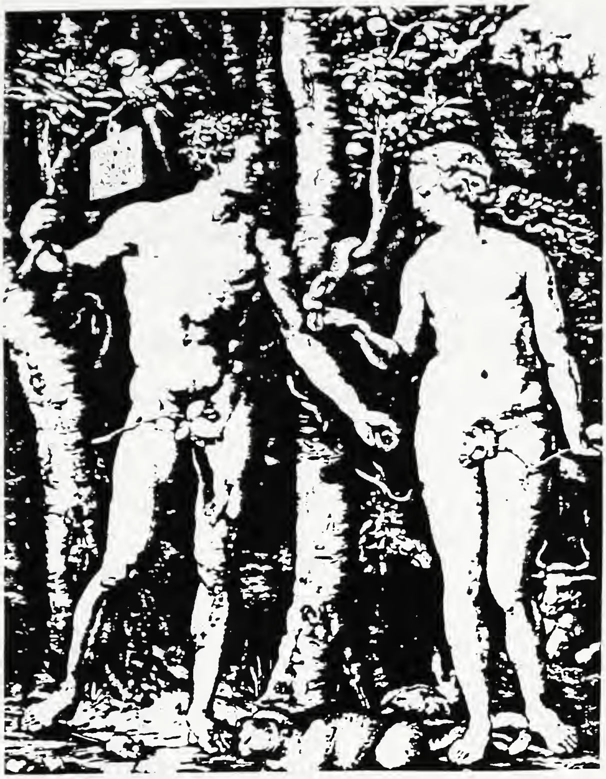

Figure 16. Albrecht Dürer, The Fall of Man

Just as Vitruvius included a canon of human proportion in his codification of Greek architectural orders, several attempts have been made "to identify human proportions with those of buildings and parts of buildings, in order to demonstrate both the architectonic 'symmetry' of the human body and the anthropomorphic vitality of architecture."[27]

"nowhere else has Dürer treated the flesh with such caressing care, using much fine dotting in the modeling, and in no previous plate has he used such a variety of textures in the conscious striving for color" (Koehler, K. 34); and it is "proudly signed with his full name in Latin, 'Albertus Durer Noricus faciebat 1504' " (E. H. Gombrich, The Story of Art [8th ed.; London: Phaidon, 1956], p. 254). See also Panofsky, op. cit., pp. 249-51.

[27] Panofsky, op. cit., p. 92.

Le Corbusier, the twentieth-century French architect, developed a system of proportion called the Modulor,[28] based on golden proportion and related to the height of a human. (Like Vitruvius, Le Corbusier assumes the navel to be the center of a man with arms outstretched; it is also taken to be the point of golden section of a standing man.) The Modulor is a composite system consisting of simple elements: the square, the double square, and the geometric series derived from the golden section of each of them (Figure 17).

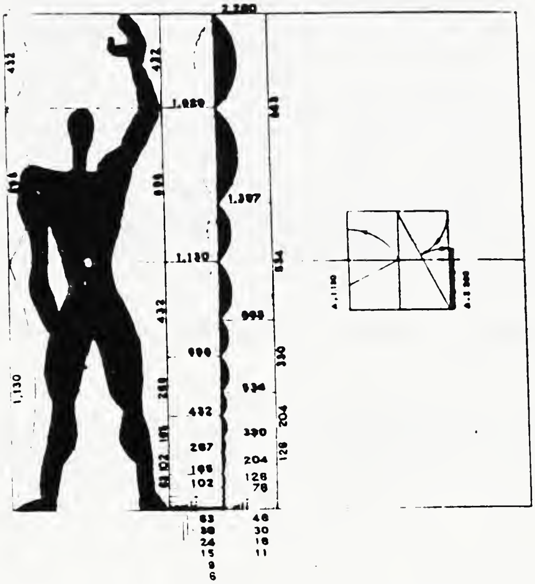

Figure 17. Le Corbusier, Le Modulor

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[28] Le Corbusier (C. E. Jeanneret-Gris), The Modulor (London: Faber and Faber, 1954; from the French, 1951).

Golden section has also been used, both in the Renaissance and during the last century, to establish the area divisions of a canvas. "The linear design of Georges Seurat's oil painting 'La Parade.' 1887–88, composed of ovals and rectangles arranged in an overall grid pattern, [is] based on the Golden Mean system of proportions" (Figure 18).[29]

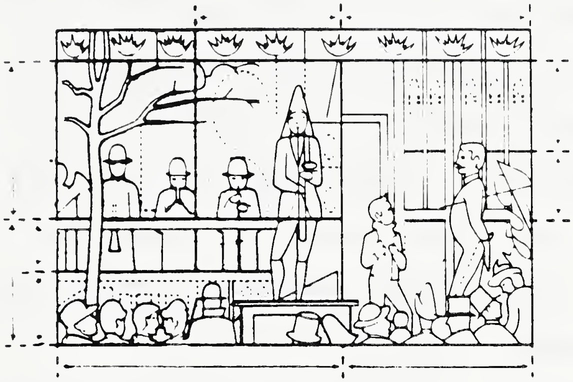

Figure 18. Area division of Seurat's La Parade

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[29] Peter D. Owen, "Art of Painting," Encyclopedia Britannica (15th ed., 1977), XIII, 870, Fig. 1.

The presence of golden proportion in nature (in phyllotaxis, growth patterns and, according to some theorists, in human proportion) and in various types, styles, and periods of art has led aestheticists to speculate and psychologists to investigate whether or not golden section has some special appeal for the human beast. The nineteenth-century aestheticist Adolph Zeising searched for "fundamental morphological laws permeating all of Nature and Art,"[30] based on a study of the proportions of the human body; he did not maintain, as is commonly held, "that the golden section holds the key to all forms of beauty . . . It was the basis for what he called 'proportionality,' representing a superior form of 'pure beauty' to 'regularity,' which is achieved by symmetry and likeness between parts. But a still higher form was 'expression,' which involves no similarity of parts or relations at all and corresponds to moderate deviations from regularity."[31]

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[30] Adolph Zeising, Neue Lehre von den Proportionen des Menschlichen Körpers . . . (Leipzig: Rudolph Weisel, 1854), title page; see also his Aesthetische Forschungen (Frankfurt: Medinger, 1855), Der goldene Schnitt (Leipzig: Engelmann, 1884).

[31] D. E. Berlyne, Aesthetics and Psychobiology (New York: Appleton-Century-Crofts, 1971), p. 222-23; see pp. 221-32 for an excellent review of the aesthetic arguments and experiments dealing with golden section, including a discussion of six hypotheses advanced to explain the importance of golden section in the visual arts.

One of the first advocates of experimental aesthetics—"an aesthetics 'from below,' meaning by that one that began with particular facts and worked its way up to generalizations"[32]—was the German physicist, philosopher and psychologist Gustave Fechner, and "data supporting the claims to attention of the golden section were among the earliest fruits"[33] of the new science. Fechner found that when subjects were asked to choose from among a variety of rectangles the one they liked best, the majority preferred the one approximating a golden rectangle (Figure 19).[34] Similar studies, including ones involving Oriental and European art works and testing Japanese and Canadian subjects, have shown varying results, but there is a tantalizing tendency for the mean result to coincide with the golden mean.[35]

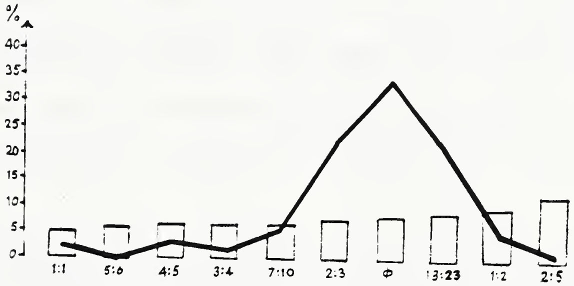

Figure 19. Fechner's graph of preferred rectangles

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[32] Ibid., p. 11.

[33] Ibid., p. 226.

[34] G. T. Fechner, "Uber die Frage des goldnen Schnitts." Archiv für die zeichnenden Künste, XI (1865), 100-112; see also his Vorschule der Aesthetic (Leipzig: Breitkopt & Härtel, 1876).

[35] Berlyne, op. cit., pp. 225, 229-30.

The most successful proponent of golden section as the basis of an aesthetic was undoubtedly Jay Hambidge, a Yale professor who developed the theory that "what the logarithmic spiral means for the structural understanding of growth in botany, as in shell formations, etc., the incommensurable root rectangles (with their sides of $1:\sqrt{2}$, $1:\sqrt{3}$, $1:\sqrt{5}$) mean for the structural understanding of Greek art and architecture"[36] (Figure 20). He argued that the proportions produced by rational number ratios (such as 1:2, 4:5) are "static," whereas those based on irrational numbers lead to "dynamic symmetry," with its suggestions of vitality, flexibility, and the "realization of nature's ideal."[37] The idea was both analyzed and popularized through the writing and teaching of Hambidge and a host of archeologists, architects, artists, and aestheticists, both defenders and detractors, with the result that the principle has been applied in all facets of commercial design, from radio cabinets to monasteries.[38] (This raises the question of whether golden proportion is popular because it is appealing, or appealing because it is popular.)[39]

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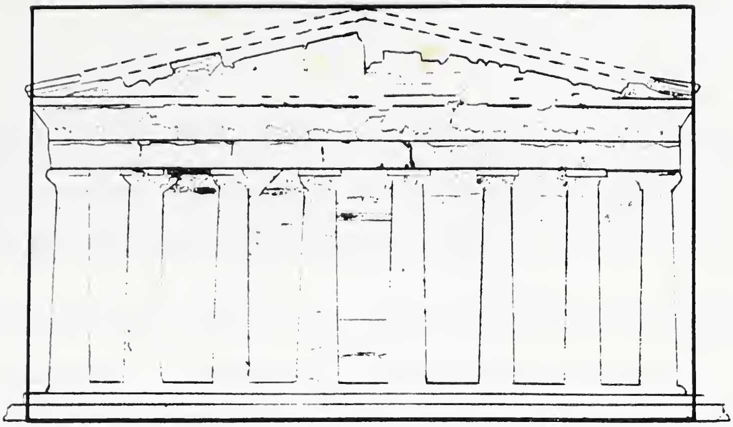
[36] Rudolf Wittkower, "The Changing Concept of Proportion," in The Visual Arts Today, ed. Gyorgy Kepes (Middletown, Conn.: Wesleyan University Press, 1960), p. 210-211. This historical outline of interest in various types of proportion is based on his paper "Systems of Proportion" (Architects' Year Book, 1955, V, 9-18).

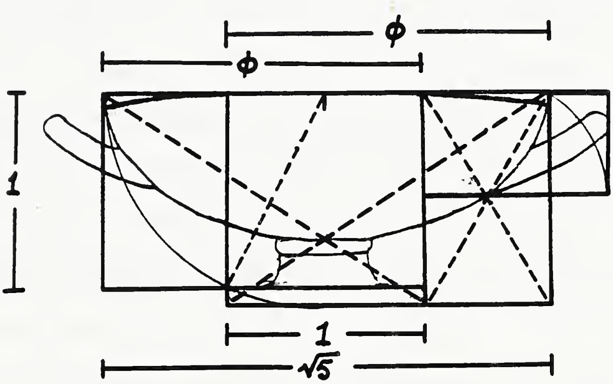
[37] Jay Hambidge, Dynamic Symmetry: The Greek Vase (New Haven, Conn.: Yale University Press, 1920),p. 142; includes R. C. Archibald's "Notes on the Logarithmic Spiral, Golden Section and the Fibonacci Series," pp. 152ff. Other works by Hambidge include Dynamic Symmetry in Composition as Used by the Artists (New York: Brentano's, 1923), The Parthenon and Other Greek Temples (New Haven: Yale University Press, 1924), The Elements of Dynamic Symmetry (New York: Brentano's 1926), and Practical Applications of Dynamic Symmetry (New Haven: Yale University Press, 1932).

[38] For example, Le Corbusier's La Tourette, a Dominican monastery in France.

[39] See Berlyne, op. cit., p. 230.



a. The Parthenon and a golden rectangle



b. A Greek drinking cup and a √5 rectangle

Figure 20. Dynamic symmetry

> Serialism is the compositional procedure by which an order of succession is established for the values appropriate to one or more parameters (components) of the musical process; these orders of succession or permutations of them are then repeated throughout the composition. 40

Musicians have always had a concern for symmetry, proportion, and organization: the reflections of inversion and retrograde motion, the rotation of retrograde inversion, and the translation of sequences and canonic and fugal imitation; the geometric proportions of bars, phrases, and periods (1, 2, 4, 8, 16, ...); the arithmetic series of overtones (A = 110 Hz., a = 220, e = 330, a' = 440, c#" = 550, ...), which determines the ratios of acoustically-correct intervals (octave = 1/2, fifth = 2/3, fourth = 3/4, major third = 4/5, ...); the temporal divisions of the rhythmic modes, mensuration, and meter; the selection and ordering of pitches by modality, tonality, and dodecaphony--all have fascinated and challenged musicians in various ages. "The musician is always counting, even when he is not aware of it."[41]

Thus, it is only natural that the principle of golden section would also be applied to music. Among the first writers on the subject was Joseph Schillinger, who urged the use of "organic forms in melody"[42] by generating melodies and scales with the Fibonacci and other recursive series.

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[40] Ernst Krenek, "Serialism," Dictionary of Contemporary Music, ed. John Vinton, p. 670.

[41] Ernő Lendvai, "Bartok und die Zahl," Melos, XXVII (1960), 327.

[42] Title of Book IV, chap. 8 of The Schillinger System of Musical Composition, ed. L. Dowling and A. Shaw (New York: Carl Fischer, 1946), p. 329.

Ernö Lendvai has written extensively[43] on his findings that the Hungarian composer Bela Bartok used golden section to partition the duration of many of his pieces. Lendvai further suggests that many of the scales Bartok used to determine melodies and chords are derived from the Fibonacci series, and that the ranges of such melodies are also controlled by golden proportion.

Contemporary German composer Karlheinz Stockhausen has used the Fibonacci series not only to determine durations, but meter signatures and the number of attacks per bar as well. His Darmstadt colleague Luigi Nono of Italy has used it primarily to serialize the duration of individual notes.

As Jonathan Kramer summarizes (in his article "The Fibonacci Series in Twentieth-century Music"), "for those composers who wish to avoid the regularity and periodicity of the series associated with traditional tonality yet wish to preserve the same degree of balance and symmetry, the Fibonacci series provides one way of reaching such a goal."[44] Kramer outlines five properties of the Fibonacci series which are relevant to its use in music,[45] several of which have been discussed above:

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[43] In addition to the article already cited, see Bartok's Style (Budapest: Zenemükiado, 1955); "Introduction aux formes et harmonies Bartokiennes" in Bartok: sa vie et son œuvre, ed. Bence Szabolcsi (Budapest: Corvina, 1956) or its German translation, "Einführung in die Formen- und Harmonienwelt Bartoks" in Bela Bartok: Weg und Werk, Schriften und Briefe, zusammengestellt von Bence Szabolcsi (Budapest: Corvina, 1957), pp. 91-137; "Duality and Synthesis in the Music of Bela Bartok," New Hungarian Quarterly, III (1962), 91-114, reprinted in Module, Proportion, Symmetry, Rhythm, ed. Gyorgy Kepes (New York: Braziller, 1966), pp. 174-93; "Bartok und der Goldene Schnitt," Osterreichische Musikzeitschrift, XXI (1966), 607-14; Bela Bartok: An Analysis of his Music (London: Kahn & Averill, 1971).

[44] Journal of Music Theory, XVII (1973), 142.

[45] Ibid., p. 114-18.

1. The additive property--"the fact that any term of the series is the sum of the next smaller two terms can be made perceptible and significant in music."
2. The golden mean property--"the sense of symmetry provided by golden mean spatial partitioning [as used in the visual arts] is, at least to some extent, carried into temporal and pitch-registral spaces." (Further, the Fibonacci series allows the use of whole numbers while still approximating the "dynamic symmetry" of Hambidges's incommensurable ratios.)
3. The proportional property--"the Fibonacci series approximates a geometric series, not, to be sure, with such simple terms as 1, 2, 4, 8, 16, ..., but with the same multiplicative property."
4. The 3:2 property--"Stockhausen claims to be interested in Fibonacci durational proportions because they approximate a geometric series of terms in the ratio of 3:2, hence providing a 'perfect fifth of duration.' " Kramer speculates "that the reason Stockhausen finds the results . . . satisfactory is not that they . . . bring pitch and duration ratios together, but rather because of the golden mean and proportional properties."
5. The relatively prime property--"Any two successive terms of the Fibonacci series are relatively prime. In other words, they have no common integer divisor other than 1. . . . Thus, on the term-to-term level a certain asymmetry or aperiodicity or slight unpredictability obtains," yet the series is still well ordered.[47]

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[47] The relatively prime property holds for all series of the form $U_n = U_{n-1} + U_{n-2}$ where the first two terms are relatively prime; see Appendix I, p. 166, for proof of this assertion.

Of the four parameters of musical sound--duration, pitch, loudness, and timbre--only the first two lend themselves readily to serialization by a recursive series.[48] As noted above, the Fibonacci series has been used in temporal space to determine not only the durations of individual notes, phrases, or sections, but the meter, tempo, and number of attacks per bar as well. More generally, there is evidence that composers tend to subdivide their works or place musically significant events at points of golden section.[49]

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[48] Kramer states that he has "generated by computer sonorities containing only Fibonacci partials with amplitudes determined by the Fibonacci series, but the results were not interesting enough to use compositionally" ("The Fibonacci Series in Twentieth-century Music," p. 141). Since loudness is not generally perceived in absolute values, nor one level heard as a sum or multiple of other levels (except perhaps in a rigorous scheme of terraced dynamics), the properties of a recursive series which would suggest its use in serialization could not easily be made perceptible or compositionally significant.

[49] Clive Pascoe, "Golden Proportion in Musical Design," concludes "that the 'pacing' (or positioning) of musically significant events, in a random sampling of compositions by fifteen 'great' composers, is frequently in coincidence with points of geometric progression based upon Golden Proportion" (p. 224). In each of fifty randomly-selected instrumental compositions, a number of musical events were identified and rated as to type and significance (high, moderate, or low structural and/or motivational significance). The entire composition was then graphed on an _a priori_ scheme of fourteen modules, each in golden proportion to the modules on either side of it; the determination of the position of each selected event was based on "seconds of performance time" (p. 83).

A recursive series may be applied in registral space in a variety of ways. First, the numbers may determine the interval (number of semitones) between successive notes of a row (Table V); due to the additive property, this is equivalent to determining the interval of each successive note of the row above a given pitch (Figure 21).

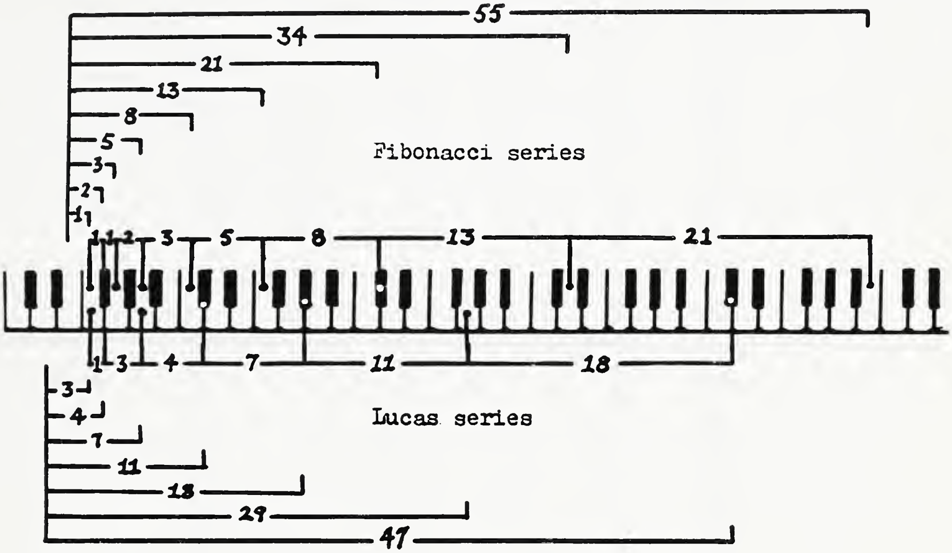

Figure 21. Pitch selection by recursive series intervals

However, if octave transpositions are allowed (as in Schillinger's "organic" melodies), the additive and proportional properties are lost, and the source of the pitch series is rendered imperceptible.[50]

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[50] Kramer concludes regarding Schillinger's pitch system that "if the resulting lines are in fact aesthetically pleasing, I suspect that the Fibonacci series is not the reason" (op. cit., p. 141).

TABLE V

Intervals Determined by the Fibonacci and Lucas Series

Series	Terms	Reduced modulo 12	Interval	+ Octaves
Fibonacci	1	1	m2	-
	2	2	M2	-
	3	3	m3	-
	5	5	P4	-
	8	8	m6	-
	13	1	m2	1
	21	9	M6	1
	34	10	m7	2
	55	7	P5	4
	89	5	P4	7
	144	0	P1	12
	233	5	P4	19
Lucas	1	1	m2	-
	3	3	m3	-
	4	4	M3	-
	7	7	P5	-
	11	11	M7	-
	18	6	A4	1
	29	5	P4	2
	47	11	M7	3
	76	4	M3	6
	123	3	m3	10
	199	7	P5	16
	322	10	m7	26

Reduction of a term by a modulus in effect determines the remainder upon division of the number by the modulus. In mathematical terms, it is determining the smallest integer congruent with the term for a given modulus, where b is congruent to a when it differs from a by a multiple of m ($a \equiv b \pmod{m}$ when $b = a + km$ for some integer k); in musical terms, it indicates the number of semitones in an interval reduced to less than one octave.

Furthermore, no twelve-tone rows can be generated by a recursive series such as the Fibonacci or Lucas series without duplicating pitches.[51] Thus, "for the Fibonacci [or similar recursive] series to have any pitch meaning it must be applied to pitches, not pitch classes."[52]

If the available recursively significant notes are thus "limited to registrally frozen pitches,"[53] and if the larger terms of the recursive series are used in order to make perceptible the golden mean property, then the resulting pitch structures are extremely restrictive in both pitch selection and range. While it is possible to compose rigorously serial pieces with such limitations,[54] other less stringent applications suggest themselves. For example, the successive pitch climaxes of a melody, section, or piece could be controlled by a recursive series, just as they are often determined by a step progression. (The series might be used in reverse order, so that the piece would quickly increase in range at the beginning, climb more gradually as the smaller terms were used, finally reaching the overall pitch climax by step.) The additive, proportional, and golden mean properties could easily be made perceptible in such a progression.

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[51] This is because in the addition of integers modulo 12—which is equivalent in musical terms to the addition of intervals allowing octave transposition—the sum of two even or two odd digits will always be even, while the sum of one even and one odd digit will be odd. Thus the terms of a recursive series of the form $U_n = U_{n-1} + U_{n-2}$ reduced modulo 12 will always be in the order:
... even, odd, odd, even, odd, odd, even, odd, odd, even, odd, odd, ...
with the result that within twelve terms at least two of the odd numbers will be duplicates, as only six different odd numbers are available.

[52] Kramer, op. cit., p. 134.

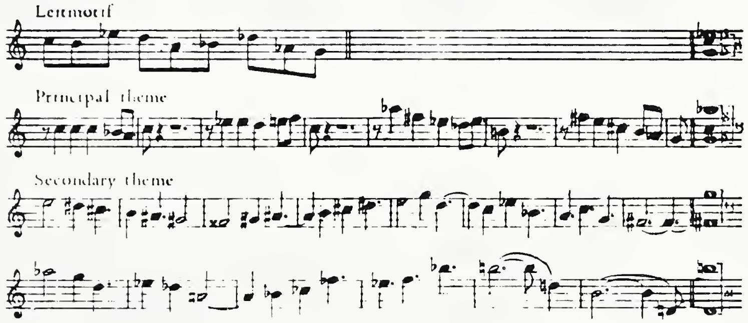
[53] Ibid.

[54] Kramer surveys several such compositions by California-based composer Don Walker (op. cit., pp. 136-41).

Ernő Lendvai has claimed that Bartok controlled the ranges of significant themes with the Fibonacci series; he cites the themes in the first movement of the Sonata for Two Pianos and Percussion:

> . . . the musical tissue may be imagined to be built up of cells, 2, 3, 5, 8, and 13 in size, with cell division following the pattern provided by the proportions of the [Fibonacci] series. . . . The opening of the leitmotif includes 8 semitones, divided by the fundamental note C into 5 + 3 [Example 1]. The principal theme comprises 13 semitones, divided by the fundamental note C into 5 + 8. The first phase of the secondary theme embraces 13 semitones, the second phase 21 semitones. Hence the various themes develop in golden-section order. 55

Example 1. First-movement themes,
Sonata for Two Pianos and Percussion



leitmotif 3 + 5 = 8
principal theme 5 + 8 = 13
secondary theme 13, 21

Other similar examples are given from the Divertimento, The Miraculous Mandarin, and the Dance Suite; in the latter two cases, the predominant interval of each successive melody, rather than its range, is seen to be governed by the Fibonacci series.

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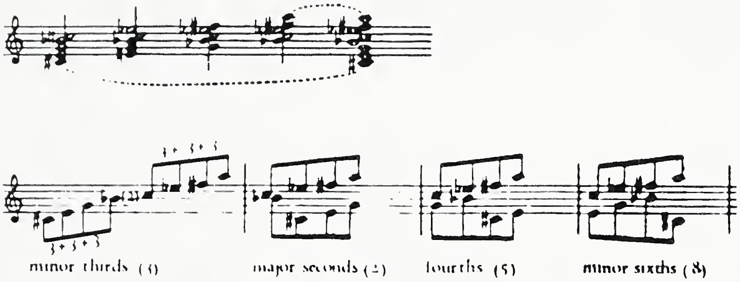
55 "Duality and Synthesis in the Music of Bela Bartok," New Hungarian Quarterly, III (1962), 95. Kramer criticizes Lendvai's claims, noting that "as with his Fibonacci duration analyses, Lendvai tries to make more of his observations than they deserve, but there is an essential truth" (op. cit., p. 134).

Lendvai also sees the influence of golden section in Bartok's harmonic architecture:

> The use of the following harmony types is, in Bartok's music, just as universal and common as the use of seventh chords in traditional music. They are constructed of nothing but "Fibonacci-cells" [Example 2]. 56

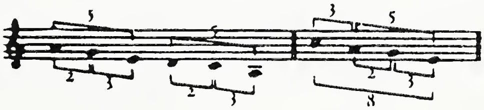
Example 2. Common harmony types in Bartok's music



This leads Lendvai to conclude:

> That golden section is not an exogenous restriction but one of the intrinsic laws of music is evidenced most convincingly by pentatony, perhaps the most ancient human sound system, which may be regarded as the purest musical conception of the principle of golden section. Pentatony, particularly the more ancient form of minor pentatony, rests on a pattern reflected by the melody steps of major second (2), minor third (3), and fourth (5) [Example 3]. 57

Example 3. Ancient form of minor pentatony



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56 "Bartok und der Goldene Schnitt," p. 612.
57 "Duality and Synthesis," p. 98.

In contrast, Kramer is led to conclude:

> Bartok may well have derived such pitch structures from the same series that he sometimes used for durational proportions, but this identity is ultimately irrelevant. It cannot be perceived nor can any consequences of it be perceived, mainly because the pitch structures use only the Fibonacci values 1, 2, 3, 5, and occasionally 8. Lendvai's examples are more likely to be heard as a stylistic predilection for avoiding interval 4 (major third) than as a combination of Fibonacci intervals 1, 2, 3, and 5. 58

While skepticism concerning golden section as an "intrinsic law of music" may be warranted, its use in music can not be dismissed even when "it cannot be perceived." As Berlyne points out:

> . . . many sources of pleasure and reward are common to the creative artist and the appreciator. But . . . the artist is also affected by certain kinds of reinforcement and satisfaction that are peculiar to his role. . . . the artist often derives intellectual pleasure from procedures that he uses in constructing his works and that are not readily apparent to anyone who simply perceives the finished product. 59

Edward A. Lowman expands on this idea in concluding his review of Lendvai's theories:

> Bartok may have had a mathematical interest in the golden mean, or he may have hit upon the Fibonacci series while consciously searching for shapes other than powers of two for his musical ideas. More likely, however, the techniques grew out of the shapes of the musical ideas themselves, just as have most new techniques throughout music history. 60

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58 Op. cit., p. 134.

59 Aesthetics and Psychobiology, p. 230.

60 "Some Striking Proportions in the Music of Bela Bartok," Fibonacci Quarterly, IX (1971), 537.

Irrespective of a composer's motivations to use golden proportion and of a listener's ability to perceive it, an analyst has a responsibility to identify such usage where it can be found, and to theorize about its origins and significance insofar as the data allow.
It is not to be expected that any work of art, let alone an artist's entire _œuvre_ or the creations of a period of history, will ever be totally understood. In fact, it is to be hoped that we will continually be confronted with art concealing art.

ARS BREVIS

There is
one art,
no more,
no less:
to do
all things
with art-
lessness.

-Piet Hein[61]

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[61] _GROOKS 1_ (Toronto: General, 1969), p. 1.

CHAPTER II

BARTOK, FOLK MUSIC, AND GOLDEN SECTION

Bela Bartok, on his unsuccessful entry in the Prix Rubinstein competition in Paris:

> Five people were candidates for the composition prize; the works of the other four were below average, mine above, but the conservative members of the jury, who subscribed to the byword "the golden mean," didn't award the prize to anyone at all. 1

Ernő Lendvai, controversial analyst of Bartok's music:

> The words of Bartok's musical language stem from the deepest layer of folk music. He himself strongly believed that every folk music of the world can finally be traced to a few primeval sources; in creating his musical idiom he was demonstrably inspired by the possibility of such a "primeval" music. Now, what is to be denoted hereafter as the "golden-section system" is simply an integration into a system of pentatonic primeval motions and primitive affinities. Bartok's most characteristic chords can be traced to simple pentatonic basic steps. Moreover, from stylistic analysis of Bartok's music I have been able to conclude that the chief feature of his chromatic technique is obedience to the laws of golden section in every element. 2

Bela Bartok, from the third of four lectures on "The New Hungarian Art Music" given at Harvard University beginning in February, 1943:

> . . . I never was concerned with general theories to be applied to the works I was going to write. Now that the greatest part of my work has already been written, certain general tendencies appear--general formulas from which theories can be deduced. But even now I would prefer to try new ways and means instead of deducing theories. 3

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1 From a letter of August 15, 1905, to Irmy Jurkovics, cited in Halsey Stevens, The Life and Music of Bela Bartok (rev. ed.; London: Oxford University Press, 1964), p. 28 (trans. from Lev. I, p. 56).

2 "Duality and Synthesis," pp. 92-93.

3 "Harvard Lectures," Bela Bartok Essays, ed. Benjamin Suchoff (New York: St. Martin's Press, 1976), p. 376.

The most noted examples of the application of the principle of golden section in music are those reported by Ernő Lendvai from the music of Bela Bartok. Among the works cited as evidencing golden proportion in some aspect--durational proportioning, melodic range, size of predominant motives, structure and spacing of typical chords, or the use of a golden-section scale--are Allegro Barbaro (1911), The Miraculous Mandarin (1919), Dance Suite ("Eastern European Symphony;" 1923), Cantata Profana (1930), Music for Strings, Percussion and Celesta (1936), Sonata for Two Pianos and Percussion (1937), Contrasts (1938), Divertimento for String Orchestra (1939), and several pieces from Mikrokosmos, Vol. 6 (1939).[4]

In light of Bartok's disparaging reference to "the golden mean" in 1905 and his disavowal of a "general theory" of composition as late as 1943, it would seem that Lendvai's claims should be taken, as Bartok was wont to say, cum grano salis.[5]

However, as John Vinton points out:

> When examining Bela Bartok's remarks about music, one should remember two things: his remarks about the folk music of Eastern Europe were the product of tens of thousands of hours of scholarly research and systematic analysis, but his remarks about his own music (although they deal with specific technical features and are supported by numerous examples) are basically the informal thinking of a composer rather than the systematic analysis of a scholar. 6

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[4] Lendvai, "Duality and Synthesis;" Edward A. Lowman, "Some Striking Proportions in the Music of Bela Bartok," Fibonacci Quarterly IX (1971), 527-28, 536-37, cites Lendvai's article in Bartok: sa vie et son œuvre; Kramer, "The Fibonacci Series in Twentieth-century Music," pp. 118-21, cites (and criticizes) both Lendvai and Lowman. adding his analysis of "Allegro Barbaro" as a juxtaposition of Fibonacci and geometric proportioning.

[5] "Harvard Lectures," pp. 370, 383.

[6] "Bartok on his own Music," Journal of the American Musicological Society, XIX (1966), 232.

In an autobiographical note written in 1921, Bartok attributes great significance to studies undertaken _after_ 1905:

> Meanwhile the magic of Richard Strauss had evaporated. A really thorough study of Liszt's _œuvre_ . . . revealed to me the true essence of composing. I began to understand the significance of the composer's work. For the future development of music his _œuvre_ seemed to me of far greater importance than that of Strauss or even Wagner.
>
> In my studies of folk-music I discovered that what we had known as Hungarian folk songs till then were more or less trivial songs by popular composers and did not contain much that was valuable. I felt an urge to go deeper into this question and set out in 1905 to collect and study Hungarian peasant music unknown until then. . . .
>
> The outcome of these studies was of decisive influence upon my work, because it freed me from the tyrannical rule of the major and minor keys. The greater part of the collected treasure, and the more valuable part, was in old ecclesiastical or old Greek modes, or based on more primitive (pentatonic) scales, and the melodies were full of most free and varied rhythmic phrases and changes of tempi, played both _rubato_ and _giusto_. It became clear to me that the old modes, which had been forgotten in our music, had lost nothing of their vigour. Their new employment made new rhythmic combinations possible. This new way of using the diatonic scale brought freedom from the rigid use of the major and minor keys, and eventually led to a new conception of the chromatic scale, every tone of which came to be considered of equal value and could be used freely and independently.
>
> . . . In 1907, at the instigation of Kodaly, I became acquainted with Debussy's work, studied it thoroughly and was greatly surprised to find in his work "pentatonic phrases" similar in character to those contained in our peasant music. I was sure these could be attributed to influences of folk music from Eastern Europe, very likely from Russia. Similar influences can be traced in Igor Strawinsky's work. It seems therefore that, in our age, modern music has developed along similar lines in countries geographically far away from each other. It has become rejuvenated under the influence of a kind of peasant music that has remained untouched by the musical creations of the last centuries.
>
> My works which, from Opus 4 onwards, tried to convey something of the development just described were received in Budapest with animosity.[7]

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[7] "The Life of Bela Bartok," Tempo, XIII (1949), 4-5; first published as "Selbstbiographie" in _Musikblätter des Anbruch_, III/5 (1921), 87-90, and as "Oneletrajzom" in _Magyar Iras_, I/2 (1921), 33-36.

This lends credence to Lendvai's claim that:

> Bartok's style took shape suddenly, virtually between one day and the next--it burst forth, to put it more accurately. About the time he composed his First String Quartet (1908) he produced a final style which, in regard to essential features, continued unchanged to his latest works. . . . It was by no means accidental that the moment when Bartok found himself, and the magic of Strauss and Reger vanished, coincided with the discovery and study of peasant music. 8

That Bartok's folk-music discoveries led him to "new rhythmic combinations" and "a new conception of the chromatic scale" (which he dubbed "polymodal chromaticism") has been acknowledged both in the passage quoted above and elsewhere in Bartok's writings.[9] However, aside from his youthful denunciation of the conservatism of the Rubinstein competition jury (as quoted above), Bartok makes no mention of the golden mean, the Fibonacci series, nor "the intrinsic laws of music," a designation which Lendvai claims for golden section.[10]

Yet the language with which Bartok discusses folk music is often reminiscent of the rhetoric used by proponents of dynamic symmetry:

> Peasant music, in the strict sense of the word, must be regarded as a natural phenomenon; the forms in which it manifests itself are due to the instinctive transforming power of a community entirely devoid of erudition. It is just as much a natural phenomenon as, for instance, the various manifestations of Nature in fauna and flora. Correspondingly it has in its individual parts an absolute artistic perfection, a perfection in miniature forms which--one might say--is equal to the perfection of a musical masterpiece of the largest proportions. It is the classical model of how to express an idea musically in the most concise form, with the greatest simplicity of means, with freshness and life, briefly yet completely and properly proportioned. 11

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8 "Duality and Synthesis," p. 92.

9 Vinton, "Bartok on his own Music," cites eight sources, grouping Bartok's remarks--largely on the influence of folk music on his musical style--under the headings "Melodic Economy," "Melodic and Rhythmic Variability," "Tonality and Modality," and "Harmonic Mannerisms."

10 "Duality and Synthesis," p. 98; see quote on p. 37 above.

11 "The Relation of Folk Song to the Development of the Art Music of Our Time," Bela Bartok Essays, pp. 321-22; first published in English in The Sackbut, I (1921), 5-11. (Italics added; "transforming

It is possible, then, that among the unenunciated discoveries Bartok made in his study of folk music (and perhaps in Liszt's and Debussy's music as well[12]) was the principle of the golden mean, applied not as a "conservative byword" but as a "natural phenomenon" and a "classical model," imparting "proper proportion" and "artistic perfection."

Nonetheless, the fact remains that Bartok did not discuss such a discovery, nor did he reject traditional musical forms in favor of new ones based solely on golden section. Indeed, as Roswitha Traimer points out:

> In the various treatises on Bartok's composition technique, his adherence to traditional forms is emphasized time and again; only Pollatsek (13) points out in an essay thereon that Bartok is the only one of the newer composers who has created new musical forms (particularly in the 3rd and 4th String Quartets), but unfortunately does not work out this proposition more precisely. 14

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power" italicized in the original.) Later in the same article, Bartok cites the works of Schoenberg as a "negative example": "He is free from all peasant influence and his complete alienation to Nature, which of course I do not regard as a blemish, is no doubt the reason why many find his work so difficult to understand" (p. 326).

[12] Roy Howat, "Debussy, Ravel and Bartok: Towards Some New Concepts of Form," Music and Letters, LVIII (1977), 285-93, notes that "Bartok all through his life used to emphasize his allegiance to Western musical tradition, and often singled out Debussy and Ravel as two composers to whom he felt particularly indebted," and goes on to "examine the possibility of their music anticipating the sort of structures that Lendvai discovered in Bartok" (p. 287), comparing Bartok's Music for Strings, Percussion and Celesta with Debussy's "Reflets dans l'eau" from Images and Ravel's "Oiseaux tristes" from Miroirs. He finds golden-section form used in all three, but asks "whether such planning is conscious or the result of an exceptionally refined instinct. None of the three composers discussed here left any direct written evidence of his technical methods in this respect, except for rather vague allusions frequently made by both Debussy and Bartok to their composition being inspired by Nature" (p. 292).

[13] "Music in Ungarn," Allgemeine Musikzeitung, LVI (1929), 840.

[14] Bela Bartoks Kompositionstechnik dargestellt an seinen sechs Steichquartetten (Regensburg: Gustav Bosse, 1956), p. 65.

Bence Szabolcsi outlines eight formal conceptions in Bartok's mature works:

1. The Beethoven sonata form
2. The Liszt sonata form
3. The three-movement form in which the third movement recapitulates the first
4. Bridge form
5. Wreath form, bound together by ritornelli and recapitulation
6. Monothematic forms
7. Classical or pre-classical concerto form
8. Combination of the Bach and Beethoven sonata form

He goes on to comment that:

> The constant influence of Bartok's great masters and ideals: Beethoven, Bach and Liszt, shines through all these form solutions (to which may be added the slow-fast formula of the rhapsodies, but principally the manifold forms of variation, as special categories, considered by Bartok to represent the deepest encounter of folk music with his own creative power), just as it shines through the composer's entire output. 15

The diverse strands set forth above are summarized and reconciled by Hungarian composer István Láng in his evaluation of "Bartok's Heritage":

> Ernő Lendvai has shown that the rule of the golden section predominates in Bartok's compositions. At the same time, the laws governing classical form are also quite evident in Bartok's works. This duality somewhat resembles the phenomenon of interference in physics. Waves in a coinciding phase, reinforce each other while those in a displaced phase weaken each other. Perhaps the particular strength of Bartok's compositions is due to the fact that these dual laws of form construction pulsate together in a common phase multiplying each other's effect many times over. 16

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15 Introductory study to Bela Bartok: His Life in Pictures (London: Boosey & Hawkes, 1964), pp. 64-66.

16 New Hungarian Quarterly, XI/39 (1970), 14.

The purpose of the present study is to discover the extent to which golden proportion is evident in the form of each movement of Bartok's six String Quartets, and to elucidate the relationship between this formal principle and traditional musical forms as used by Bartok. The quartets are particularly appropriate for this type of analysis because they can be seen as a longitudinal survey of Bartok's style periods, since they date from 1908, 1915-17, 1927, 1928, 1934, and 1939.[17] Moreover, as Abraham and others have attested, "the quartets represent Bartok's best or at any rate most serious work at each period."[18] Thus it is to be assumed that the compositional techniques used in the String Quartets are representative of those in other works of the same period; any difference would presumably lie in the heightened intensity of their application. While generalizations applicable to Bartok's entire *œuvre* are precluded, trends can be outlined and predictions made which may be found to have wider relevance by later, more extensive studies.

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[17] The first sketches of the first quartet probably date from 1907; it was completed on Jan. 27, 1909 (György Kroo, *A Guide to Bartok*, trans. Ruth Pataki and Maria Steiner, rev. Elisabeth West [Budapest: Corvina, 1974], p. 45). Kodaly lists an earlier, unpublished quartet of 1899 in his article on Bartok in *La Revue musicale*, II/3 (1921). An eight-bar sketch for a seventh quartet remained at his death (Colin Mason, "Bela Bartok," *Grove's Dictionary of Music and Musicians* [5th ed., ed. Eric Blom, 1954], I, 476).

[18] Gerald Abraham, "The Bartok of the Quartets," *Music and Letters*, XXVI (1945), 185; see also Halsey Stevens, *The Life and Music of Bela Bartok*, p. 170.

The method used began with a comparison of formal analyses of each quartet by various writers, to determine the formal outlines and points of subdivision of each movement. The relative dimensions of each section thus isolated were then analyzed mathematically, with a view to identifying those exhibiting golden proportion. The results of this analysis have been graphed to show in a visual summary the formal scheme outlined by each writer and the significant instances of golden proportion discovered therein.

The results indicate that, in determining form in his string quartets, Bartok used golden proportion consciously, concertedly, and consistently: consistently since it is used in each movement of every quartet, from Opus 7, sketched just two years after his discovery of folk music, to his last quartet, which was also his last work to be written in Europe; concertedly because several systems of golden proportion often interlock, overlap, or are otherwise extended or related, together proportioning each movement in unexpected but harmonious combinations; consciously by the fact that golden whole-number series are used exactly, integrated systems are used in a contrived manner, and the Fibonacci series, both larger and smaller terms, is used in an exposed way.

The overall results of this study are presented at the conclusion of this chapter in a series of graphs. The accompanying notes on the graphs will assist in the understanding and interpretation of many of the details of the diagrams, but it is hoped that a general impression of the results can be gained by a brief perusal of these visual summaries.

In Chapter III, the various types and systems of golden proportion as applied by Bartok are discussed in more detail, from simple golden section through hierarchies to complex and unexpected combinations. Chapter IV presents the theory that several formal structures are the result of the insertion of additional sections into an earlier draft. Chapter V discusses the possibility that these proportional systems occurred by coincidence or through intuitive genius, concluding that the use of the Fibonacci series suggests their intentional creation. Chapter VI summarizes the conclusions drawn from this study, and presents several other symmetrical and proportional systems which Bartok may have used in conjunction with golden section. The two Appendices present a summary of formulae relating to simple recursive series, especially the Fibonacci and Lucas series.

It was found, in contrast to what would normally be anticipated, that significant proportional relationships were identified when dimensions were measured metrically--by counting bars or beats--rather than temporally--by counting seconds of performance time. Furthermore, with one significant exception, golden proportions appear to be based on numbers of bars, and not beats, in spite of changing meters and tempos. There are three supportable hypotheses which may account for these findings.

First, Bartok may have assumed that perception of the flow of time in music is influenced or even determined by metric organization. Roy Howat, citing Ernő Lendvai, advances this hypothesis, arguing that "the measurement of proportion proceeds by the inner pulse of the music, thus absorbing fluctuations of tempo such as accelerandi and ritardandi, rather than by time as measured on a clock. This is quite reasonable, as Lendvai points out, since the involved listener does not look at his watch throughout a performance, but has a much more subjective experience of time, related to the flow of the music."[19]

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[19] "Debussy, Ravel and Bartok," p. 292.

However, as Howat admits, "even then inconsistencies arise."[20] While small and gradual fluctuations of tempo may be absorbed, surely abrupt tempo changes will be perceived. In this connection, John Vinton's comments regarding tempo in the first movement of the Sixth Quartet are enlightening:

> Bartok's sensitivity to *tempo* is amply illustrated by his frequent metronome markings and duration timings. His colouristic treatment of tonality undermines the structural support that harmony could have provided through modulations away from and back to a tonic. His *tempo* changes, however, by moving away from and back to a *Tempo Primo*, reinforce this weakened structure. The first movement aligns a *tempo* change with every important formal articulation . . . 21

This concept of tempo as a formal device and not merely a determinant of duration has implications for the discussion of the perceptibility of proportional relationships in Bartok's music. If the changes in tempo are relatively small, yet still perceptible, formal divisions will be articulated without drastically altering the perceived duration of each section. For example, the predominant meter of the movement in question is 6/8; a thirty-five-bar section at ♩.=140 would last thirty seconds, while a similar thirty-five-bar section at ♩.=120 would last thirty-five seconds. A listener primarily concerned with content rather than duration—with what is said rather than how long it takes to say it--would perceive these sections as being of equal length (or perhaps weight).

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20 Ibid.

21 "New Light on Bartok's sixth Quartet," *Music Review*, XXV (1964), 226.

Even more problematic are changes in meter. Bartok describes one of the three kinds of rhythm prevalent in Eastern European rural music as "the more or less rigid rhythm, with regularly set bars, generally in 2/4 time. In certain types, change of measure may occur--which leads in some cases to seemingly complicated rhythms." He admits that "what mostly interested us in the rigid rhythm kind were the changes of measure. I had fully exploited these possibilities already in my earliest works, and later perhaps even with some exaggeration." However, while 3/4, 5/8, or 7/8 time may be used alternately with 2/4 time, Bartok maintains that "the difference between them and the regular 2/4 is not essential; it is rather a derivative difference. In fact, 5/8 can be explained as a doubling of one of the eighths in a 2/4 measure, and 7/8 as a doubling of one of the eighths in a 3/4 measure."[22]

Thus changes in meter may be heard as momentary deviations from a metric norm, variations on a meter, "a rhythmic equivalent," as Vinton puts it, "of the chromatic inflections that continually colour passages."[23] This echoes Bartok's laudatory description of "a plain old rustic melody" which "sounds incomprehensibly modern" due to its use of modes and "the freest of rhythm; not the hackneyed sequence of but one kind of measure, but a rubato recital with the strangest coloratures—sometimes four or five variations in a measure within one short melody. Truly difficult to understand!"[24]

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22 "Harvard Lectures," pp. 383, 386, 391-92; the last phrase was emended by the editor, as "Bartok indicates 7/8 as a trebling of one of the eighths in a 4/4 measure. See, however, his discussion on Bulgarian rhythm in Essay No. 9 (p. 41)" (n. 1, p. 392).

23 "New Light," p. 227.

24 "The Peasant Music of Hungary," The Musical Courier, Sept. 12, 1931. p. 6, cited in Vinton, "Bartok on his own Music." p. 235-36.

Howat rationalizes his measurement of Ravel's "Oiseaux tristes" in beats but the fugal first movement of Bartok's Music for Strings, Percussion and Celesta in bars--although both have meter changes--on the principle that "the best approach is probably to seek the way in which the musical pulse is most clearly manifested. In the Bartok fugue the quaver movement is ill-defined, while the movement breathes very clearly phrase by phrase, of which there is one to each bar all through the movement. Listening to the piece, it is not difficult to feel where all the bar-lines fall, whereas this is much more difficult in 'Oiseaux tristes', which does however have a much more discernible crotchet pulse."[25]

However, this does not hold true throughout the quartets, which contain numerous passages in which the changes in meter are difficult or impossible to feel--the rhapsodic, Hungarian theme in the seconda parte of the Third Quartet and the Meno vivo in the coda of the same work are even notated with beams crossing the bar lines of changing meters.

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[25] "Debussy, Ravel and Bartok," p. 292.

The second movement of the Fifth Quartet is the only one in which proportional relationships could be identified as measured in beats; not only are the overall movement and the first section partitioned in golden proportion to the nearest beat, but the movement is laid out symmetrically, with the first and second sections each of eighty-two beats and the final section exactly half as long at forty-one beats (Figure 22).

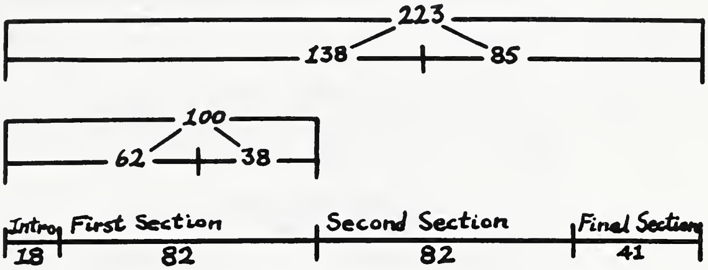

Figure 22. Proportion as measured in beats in Quartet V, movement 2

Furthermore, this is the only quartet in which half-bar lengths appeared to be significant, both notationally and proportionally. Half-bar lines are notated in bars 10 and 46 of the second movement, and some rehearsal letters are placed at the center of bars in both the first and the second movements. (Rehearsal letters are precisely placed in this quartet, most over the bar line, but many over the note or rest which begins a new section.) The entire second movement divides by golden section at one of these rehearsal letters, and the larger portion is multiply subdivided by golden section, in each case to the nearest half-bar.[26]

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[26] For a more detailed discussion of this system, see Chapter III, p. 94; see also Figure 42, p. 77.

While it is to be anticipated that a section which is in exact golden proportion as measured in beats would also be in golden proportion as measured in bars, in this case the points of golden section do not coincide: the dividing point of the system measured in bars is at rehearsal letter A, subdividing the first section into two parts; the golden mean determined in beats marks the turning point in volume of the first section and the first use of the theme in inversion (Figure 23).

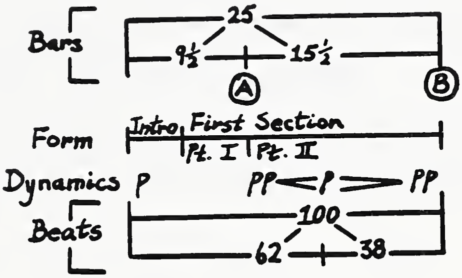


Figure 23. Proportion as measured in bars and beats in Quartet V, movement 2

Thus it appears that Bartok purposefully integrated a proportional design based on beats with another measured in bars, each serving a different function. The reason for his concern with beats in this movement and no other may lie in the fact that this is also the slowest movement of the quartets: ♩=40-38.

In light of the foregoing, the principle advanced by Howat might best be revised to state that "the best approach is probably to seek the way in which the composer conceived of the musical pulse." As Edward T. Cone has pointed out, for Baroque composers the primary metric unit was the beat; for Classic composers, the measure; and for

Romantic composers, the hypermeasure.[27] A composer like Stravinsky, who saw rhythmic accuracy as the key to interpreting his works, was no doubt thinking in terms of a metronomic pulse while composing. Bartok, on the other hand, who used tempo as a formal device, who saw the difference between meters as derivative rather than essential, and who was concerned not to write a "hackneyed sequence of but one kind of measure," probably thought in terms of bars, sensing the downbeat as the basic tactus of his music.

Evidence in support of this conclusion is offered by Robert Wykes' description of his brief examination of the original manuscript of *Music for Strings, Percussion and Celesta*:

> For the first 83 measures of the fourth movement, Bartok indicated the phrase and period length by means of Roman numerals. The phrase groups and the Roman numerals assigned to them are as follows:
>
> Phrase Group: 8 - 8 - 10 - 6 - 6
> Numerals: I II III IV VI
>
> Phrase Group: 15 (5-4-4-2) - 5 - 5 - 10 - 10 (4-4-2)
> Numerals: VII VIII IX X XII
>
> Group VII contains the same material as I with the addition of a 5-measure transition at its beginning and a 2-measure transition at its closing. Groups VIII, IX, and X are new material, and bear the same relationship to VII as III, IV, and VI to I, i.e., subsidiary ideas to the main theme. At XII is the return of a slightly altered version of the theme of I. Also, they contain almost exactly the same number of measures.
>
> At this point in the score is found the following notation: 11 plus (+) 26 = 37 sov.. A more adept student of the Hungarian language might find a different interpretation of the abbreviation sov.: however, I have taken it to be the Hungarian word *sovardas*, and meaning "the number of measures to be used in the next section," and that Bartok used this manner of sketching out the music to come and to set up its structural balance. 28

---

[27] *Musical Form and Musical Performance* (New York: Norton, 1968), pp. 66, 72, 79.

[28] "Structural elements in selected works of Bartok" (M.Mus. thesis; Eastman School of Music, U. of Rochester, 1949), pp. 11-12.

This not only leads to the conclusion that Bartok worked in terms of numbers of bars but suggests the second hypothesis of why he did so: he may have sketched out the number of bars in each section in advance of its composition as a compositional aid. With the approximate dimensions of the work predetermined, he could then fill each part with appropriate music, confident that the overall structure was sound and balanced. As Howat points out, "Formal planning is only one facet of the technique of composition, but one which Ravel nevertheless considered to be of fundamental importance, as is evident from his otherwise enigmatic remark to Maurice Delage that 'My Trio is finished. I only need the themes for it.' "[29]

Evidence attesting to the validity of this hypothesis is presented in Chapter IV, in which it is suggested, for example, that the present structure of the second movement of the Fourth Quartet is the result of additions made to an initial draft, the latter displaying golden proportion in its overall outline while the final form does not. Had Bartok been primarily concerned with the perceptibility of proportional relationships, he would have revised the whole movement or chosen additions of an appropriate length to ensure the golden sectioning of the expanded form.

---

[29] "Debussy, Ravel and Bartok," p. 292, citing quote from H. H. Stuckenschmidt, Maurice Ravel: Variations on his Life and Work (London, 1969), p. 149.

The final hypothesis is that suggested by Berlyne, as quoted in Chapter I (p. 38): that Bartok may have used golden section for his own satisfaction. Again, since perceptibility would not be the primary concern, the fact that he manipulated metric lengths rather than temporal spans would not be perplexing. Furthermore, this would explain the existence of elaborate systems of golden sectioning, such as those outlined in Chapter III, which could hardly have been meant for audible perception.

In fact, one could speculate that a possible source of satisfaction for Bartok was using the golden mean in ways that the 1905 _Prix Rubinstein_ jury could never have imagined!

NOTES ON THE GRAPHS

Each of the following graphs consists of a scale line, a graphic representation of each available analysis of the movement under consideration, and an indication of the significant golden proportional relationships found. The analyses graphed (and denoted by an asterisk in the Bibliography) are those which make specific reference to the formal divisions of the movement analyzed, as opposed to those which treat the quartets in more general terms or which deal primarily with thematic analysis.

The scale line across the bottom varies in scale depending on the length of the movement graphed. Rehearsal figures as given in the score are indicated above the line. For the first three quartets, these are rehearsal numbers, each placed above a bar line. Quartets IV and V have bar numbers indicated every five bars, which are thus of no significance in revealing structure as the composer saw it; the Fifth Quartet, however, also includes precisely-placed rehearsal letters. The Sixth Quartet has selected bar numbers indicated, these being of significance as they are not regularly spaced but were apparently chosen depending on the musical content of each passage. Timings are indicated where specified in the score.

Beneath the scale line are index numbers which are derived from bar numbers. Since a bar number refers not to a point (such as a bar line) but to a length (the entire bar), bars are counted inclusively; that is, the span "from bar 4 to bar 7" is not (7 - 4 =) three bars, but four bars, as bar 7 is included (Figure 24).

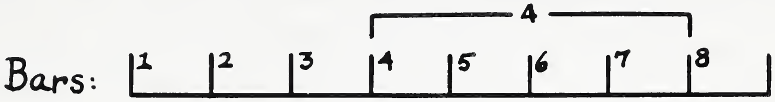


Figure 24. Inclusive counting of bars

Similarly, since the golden mean of eight is five, an eight-bar passage divides by golden section after the fifth bar (i.e., at the point at which five of the eight bars have elapsed), not at the fifth bar (which would in this case be division at the geometric center, after four bars; Figure 25).

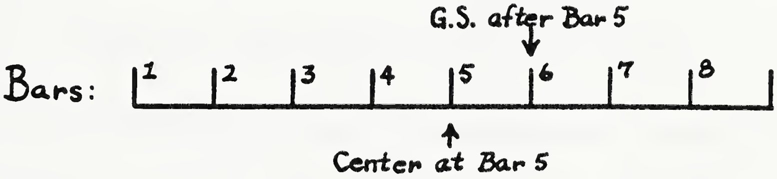


Figure 25. Golden section related to number of bars elapsed

In order to simplify the measurement of passages in bar lengths (or number of bars elapsed), it is helpful to use index numbers which refer to a point (in this case, the bar line) rather than to a unit; this allows exclusive counting (used for measurement along a continuum) rather than inclusive counting. The relationship to bar numbers is a simple one: the start of the piece is considered the origin or zero point, and each bar number is thought of as shifted to the end of the bar, marking the next bar line, and thus indicating the number of bars elapsed. Now the measurement of a span is by simple subtraction, and the golden mean is at the point marked by the index number (Figure 26).

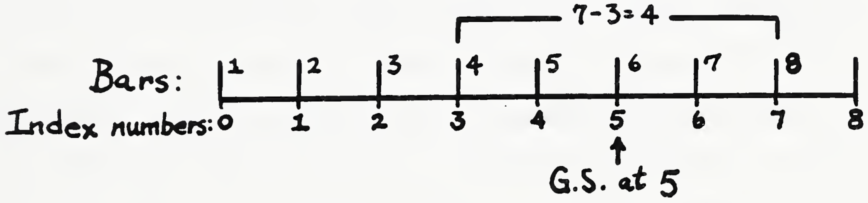


Figure 26. Index numbers related to bar numbers

In the Third Quartet, in which the movements are attacca and interrelated, two sets of index numbers are given: the upper set applies to each part (Prima parte, etc.), the lower set to the whole quartet (numbered from beginning to end). In the first three movements of the Sixth Quartet, the cyclic theme (Mesto) is numbered separately from the movement proper.

Above the scale line are graphed the analyses of the movement in question by various analysts. As primary source material, we have in addition to Bartok's own analyses of the Fourth and Fifth Quartets[30] only the following statement regarding the Second Quartet:

> The first movement is in normal sonata form, the second is rondo-like with a kind of development in the middle; the last is the most difficult to define; it is, when all is said and done, a somewhat expanded A-B-A-form. 31

The two most detailed analyses which are available for every quartet are those by Thomason and Walker.[32] The former is a motivic analysis, and the labels assigned to the various motives are indicated on the graphs in quotation marks (with "variant" abbreviated as "var"); the main tonalities are identified in capital letters below, with a question mark indicating those passages which Thomason regarded as indeterminate in tonality. Walker's thematic analysis classifies material as either thematic ("Th") or episodic ("Ep"); themes are identified by Roman numeral or as introductory ("Intro" or "Int"). "I + II" denotes a vertical combination of themes, "I/II," a melodic combination, and "I, II," the alternate use of themes; "alt" is the abbreviation for "altered," "inv" for "inverted," "var" for "varied," and "ost" for "ostinato." The Thomason and Walker studies should be consulted to identify the motives and themes labelled.

---

[30] Bela Bartok Essays, pp. 412-15.

[31] Quoted from "one of his letters (1936)" in Bence Szabolcsi, Introductory study to Bela Bartok: His Life in Pictures, p. 64.

[32] Leta Nelle Thomason, "Structural Significance of the Motive in the String Quartets of Bela Bartok" (Ph. D. dissertation, Michigan State University, 1965); Mark Fesler Walker, "Thematic, Formal, and Tonal Structure of the Bartok String Quartets" (Ph. D. dissertation, Indiana University, 1955).

Clive Pascoe's analyses of the first movement and Finale of the Fifth Quartet[33] include his specification of the degree and type of significance of each event identified in the analyses, indicated by one, two, or three "+," "-." or "x" signs placed vertically above the point of the event, signifying low, moderate, or high "structural," "motivational," or "structural and motivational" significance respectively. Pascoe's graphs of his results, which are based on seconds of performance time and superimposed on a scheme of fourteen golden modules, are included as Figure 46.[34]

In the top portion of each graph, the length of each passage which is in golden proportion to another is indicated in bar-lengths (except for Quartet V, movement 2, for which beats are also given,) with an arrow showing the extent of the span. Each length divided by golden section is related to its modules by angled lines between the arrows. A passage which is not part of a golden system but which is of a significant length is indicated by a dotted-line arrow, the length given in parentheses (Figure 27). Most golden systems are discussed in the chapters which follow.

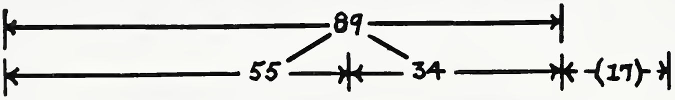


Figure 27. A span, its golden modules, and a non-golden length

---

[33] "Golden Proportion in Musical Design," pp. 128-32.
[34] Ibid., p. 252. The numbers "7" and "8" mark the points of negative and positive golden section, respectively, of the whole movement; moving outward from these points, each number marks the point of golden section of the span from the end-point to the previous number. "The dotted lines on either side of G7 and G8 indicate a 'range' of plus or minus three percent of the total length. This range suggests an 'area of influence' of G7 and G8" (p. 86).

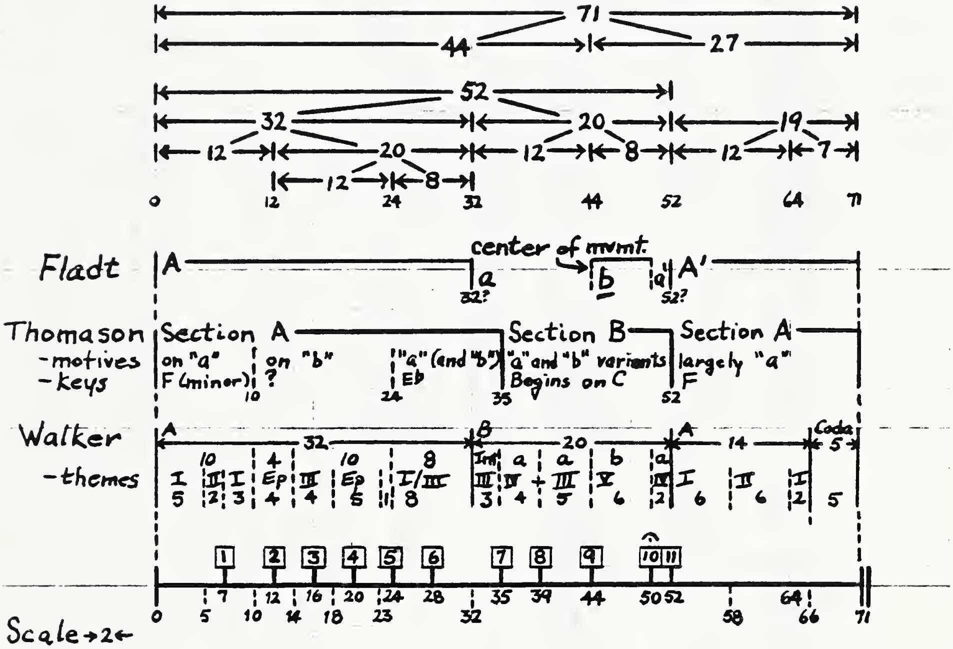

71
44
27
52
32
20
19
12
20
12
8
12
7
12
8
0
12
24
32
44
52
64
71
Fladt
A
center of mvmt.
a
32?
b
a
A'
52?
Thomason
-motives
-keys
Section A
on "a"
F (minor)
10
on "b"
?
"a" (and "b")
Eb
24
Section B
"a" and "b" variants
Begins on C
35
Section A
largely "a"
F
52
Walker
-themes
A
32
B
20
A
14
Coda
5
10
4
10
8
I 5
II 2
I 3
Ep 4
III 4
Ep 5
I/III 8
Int III 3
a IV 4
+
a III 5
b V 6
a IV 2
I 6
II 6
I 2
5
1
2
3
4
5
6
7
8
9
10
11
7
12
16
20
24
28
35
39
44
50
52
64
0
5
10
14
18
23
32
58
66
71
Scale→2←

Figure 28. Quartet I, movement 1

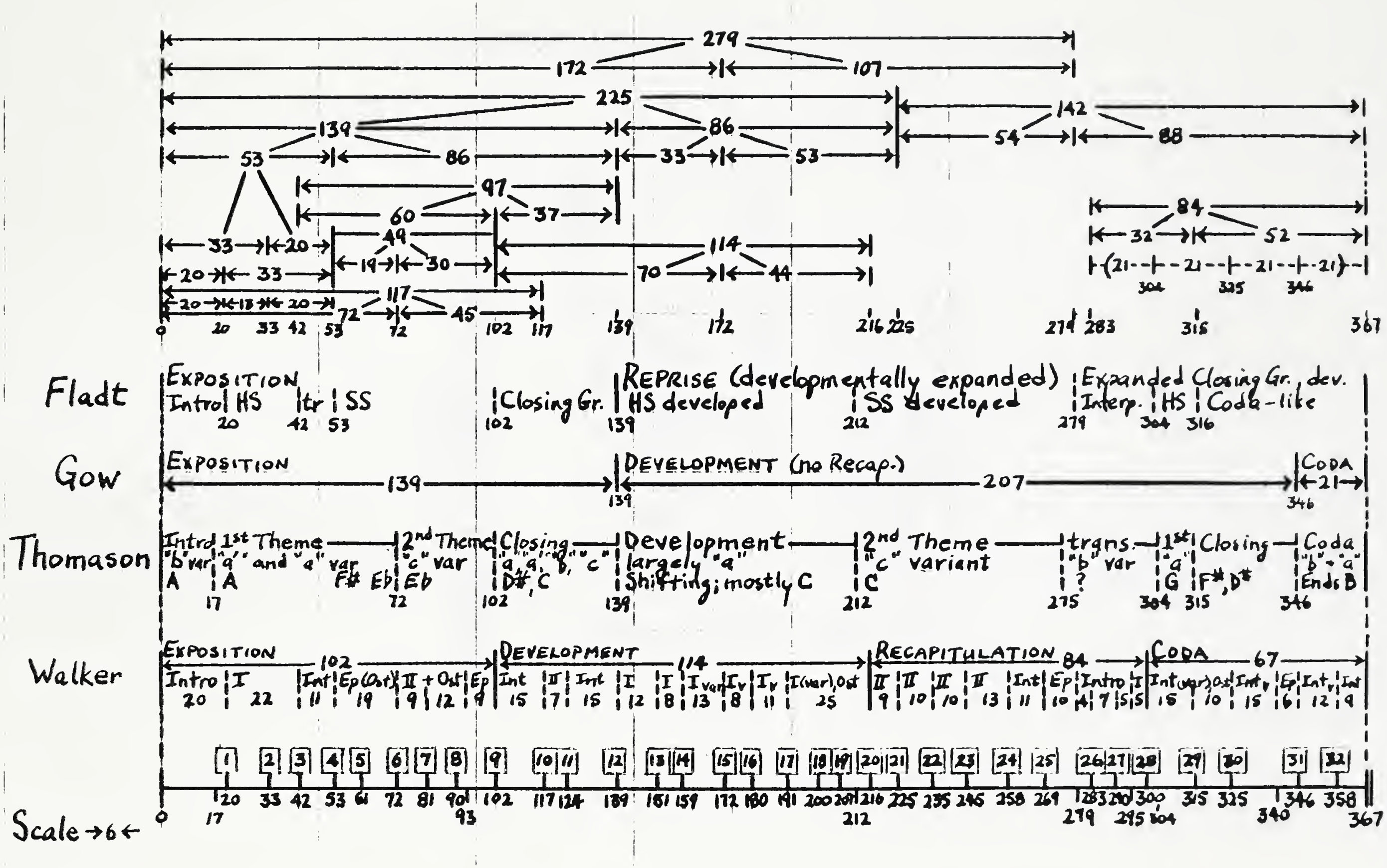

279
172
107
225
142
139
86
54
88
53
86
33
53
97
60
37
84
32
52
33
20
49
19
30
114
70
44
(21 - 21 - 21 - 21)
117
72
45
Fladt
EXPOSITION
Intro
HS
tr
SS
Closing Gr.
REPRISE (developmentally expanded)
HS developed
SS developed
Expanded Closing Gr. dev.
Interp.
HS
Coda-like
Gow
EXPOSITION
139
DEVELOPMENT (no Recap.)
207
CODA
21
Thomason
Intro "b" var A
1st Theme "a" and "a" var F# Eb
2nd Theme "c" var Eb
Closing "a", "a", "b", "c" D#, C
Development largely "a" Shifting; mostly C
2nd Theme "c" variant C
trans. "b" var ?
1st "a" G
Closing F#, D#
Coda "b" - "a" Ends B
Walker
EXPOSITION
102
DEVELOPMENT
114
RECAPITULATION
84
CODA
67
Scale → 6 ←

Figure 29. Quartet I, movement 2

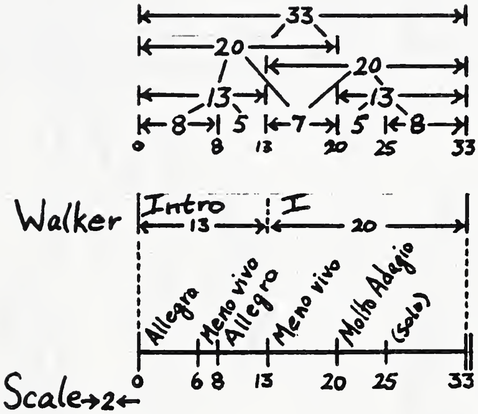

33
20
20
13
13
8
5
7
5
8
0
8
13
20
25
33
Walker
Intro
I
13
20
Allegro
Meno vivo
Allegro
Meno vivo
Molto Adagio
(solo)
Scale→2←
0
6
8
13
20
25
33

Figure 30. Quartet I, Introduzione to movement 3

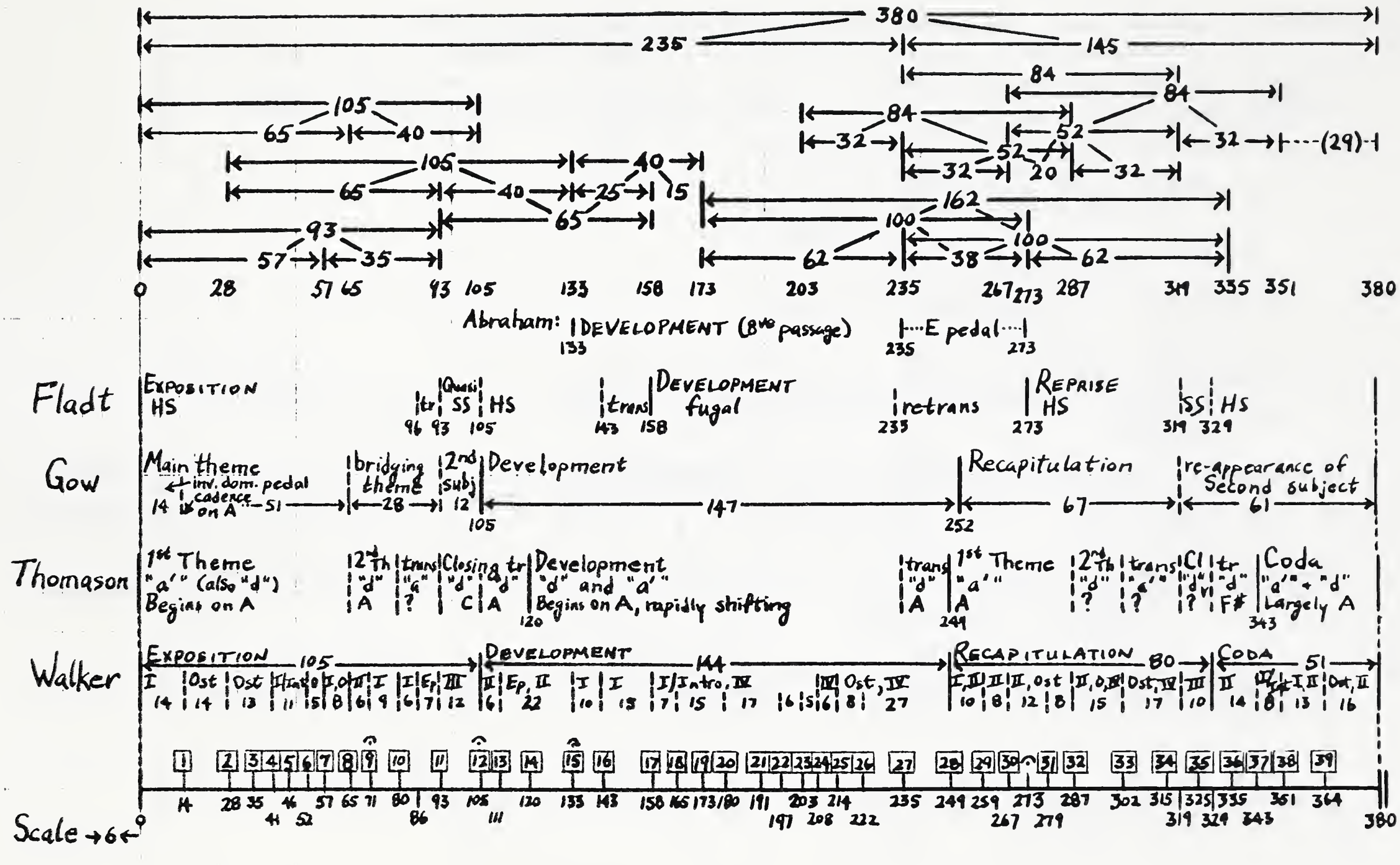

380
235
145
105
65
40
93
57
35
84
32
52
20
162
100
62
38
(29)
0 28 57 65 93 105 133 158 173 203 235 267 273 287 319 335 351 380
Abraham: DEVELOPMENT (8va passage) 133
E pedal 235 273
Fladt
EXPOSITION HS
tr 96
Quasi SS 93
HS 105
trans 143
DEVELOPMENT fugal 158
retrans 235
REPRISE HS 273
SS 319
HS 329
Gow
Main theme
inv. dom. pedal
cadence on A
14
51
bridging theme 28
2nd subj 12
Development 105
147
Recapitulation 252
67
re-appearance of Second subject
61
Thomason
1st Theme "a'" (also "d") Begins on A
2nd Th "d" A
trans "a" ?
Closing "d" C
tr "d" A
Development "d" and "a'" Begins on A, rapidly shifting 120
trans "d" A
1st Theme "a'" A 249
2nd Th "d" ?
trans "a'" ?
Cl "d" ?
tr "d" F#
Coda "a'" + "d" Largely A 343
Walker
EXPOSITION 105
DEVELOPMENT 144
RECAPITULATION 80
CODA 51
I
Ost
Intro
Ep
Scale → 6 ←

Figure 31. Quartet I, movement 3

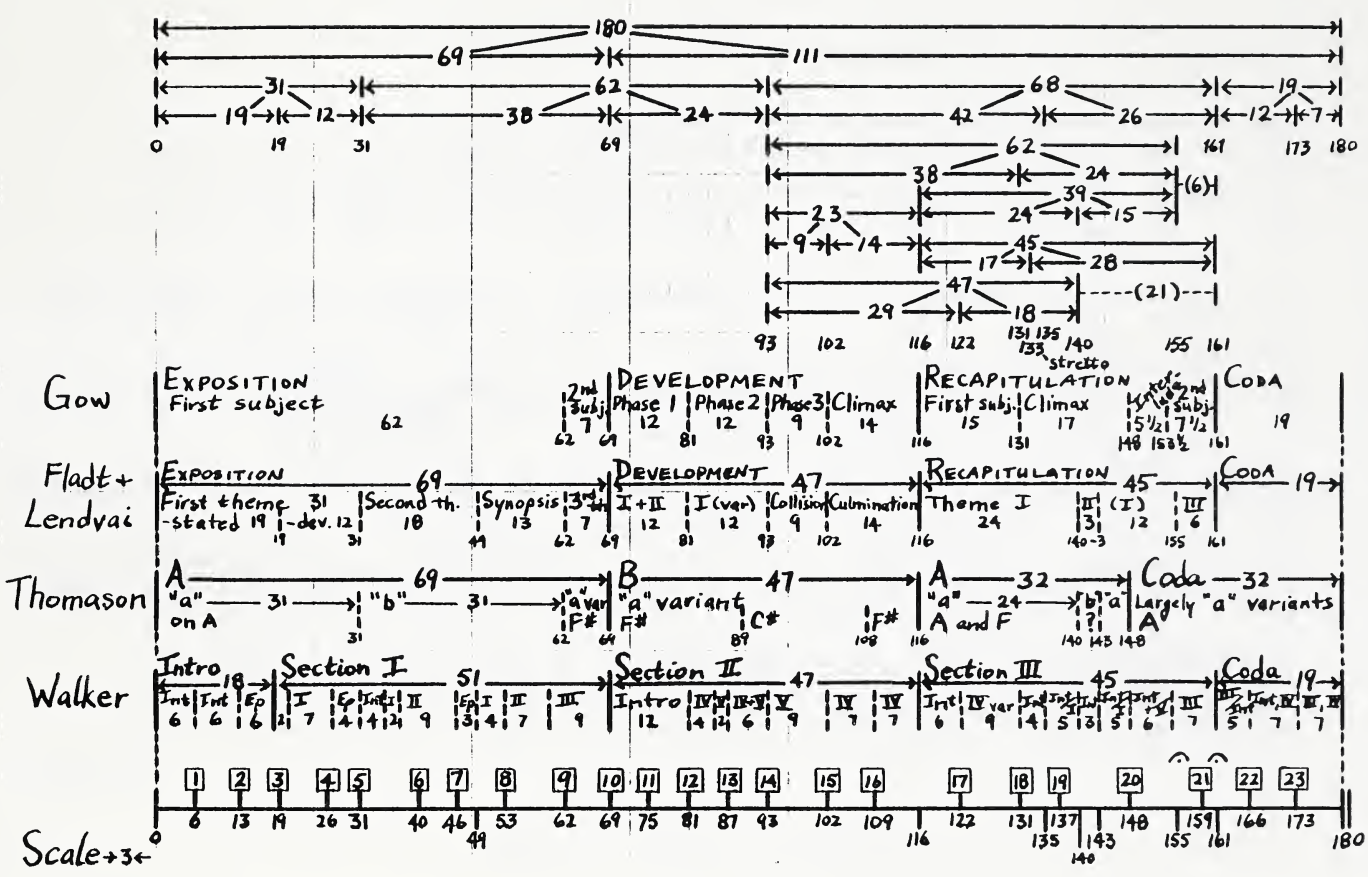

Gow
Fladt + Lendvai
Thomason
Walker
Scale ÷ 3 ←
180
69
111
31
62
68
19
12
38
24
42
26
7
0
19
31
69
161
173
180
EXPOSITION
First subject
2nd subj.
DEVELOPMENT
Phase 1
Phase 2
Phase 3
Climax
RECAPITULATION
First subj.
Climax
stretto
2nd subj.
CODA
EXPOSITION
First theme -stated
-dev.
Second th.
Synopsis
3rd th.
DEVELOPMENT
I + II
I (var)
Collision
Culmination
RECAPITULATION
Theme I
CODA
A
"a" on A
"b"
"a" var F♯
B
"a" variant F♯
C♯
F♯
A
"a" A and F
b
a
Coda
Largely "a" variants
Intro
Int
Ep
Section I
Section II
Intro
Section III
Coda
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
21
22
23
6
13
19
26
31
40
46
49
53
62
69
75
81
87
93
102
109
116
122
131
135
137
140
143
148
155
159
161
166
173
180

Figure 32. Quartet II, movement 1

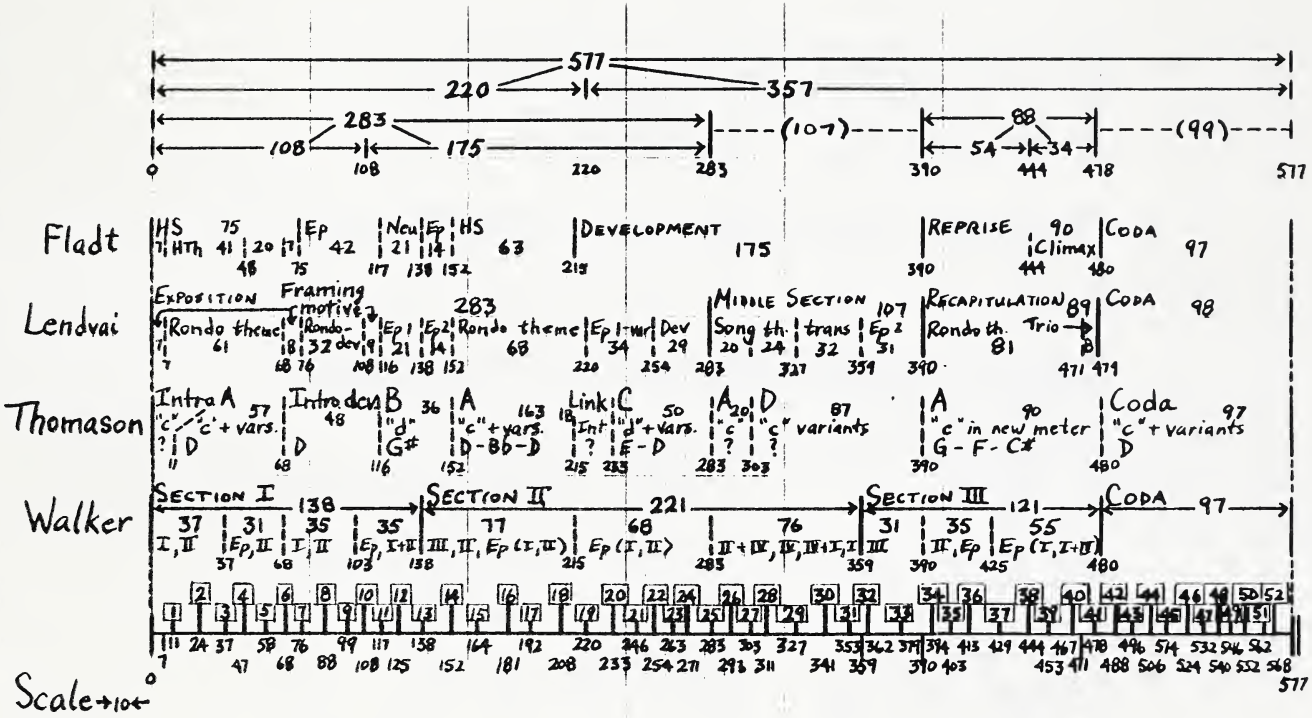

577
220
357
283
108
175
(107)
88
54
34
(99)
0
108
220
283
390
444
478
577
Fladt
HS
HTh
Ep
Neu
HS
63
DEVELOPMENT
175
REPRISE
90
Climax
CODA
97
Lendvai
EXPOSITION
Framing motive
Rondo theme
Rondo-dev
Ep 1
Ep 2
283
Rondo theme
68
Ep I-var
Dev
MIDDLE SECTION
107
Song th.
trans
Ep 2
RECAPITULATION
89
Rondo th.
81
Trio
CODA
98
Thomason
Intro A
57
"c" + vars.
Intro dev B
48
36
"d"
G#
A
163
"c" + vars.
D-Bb-D
Link
Int
C
50
"d" + vars.
E-D
A
20
D
87
"c" variants
A
90
"c" in new meter
G - F - C#
Coda
97
"c" + variants
D
Walker
SECTION I
138
SECTION II
221
SECTION III
121
CODA
97
Scale +10+

Figure 33. Quartet II, movement 2

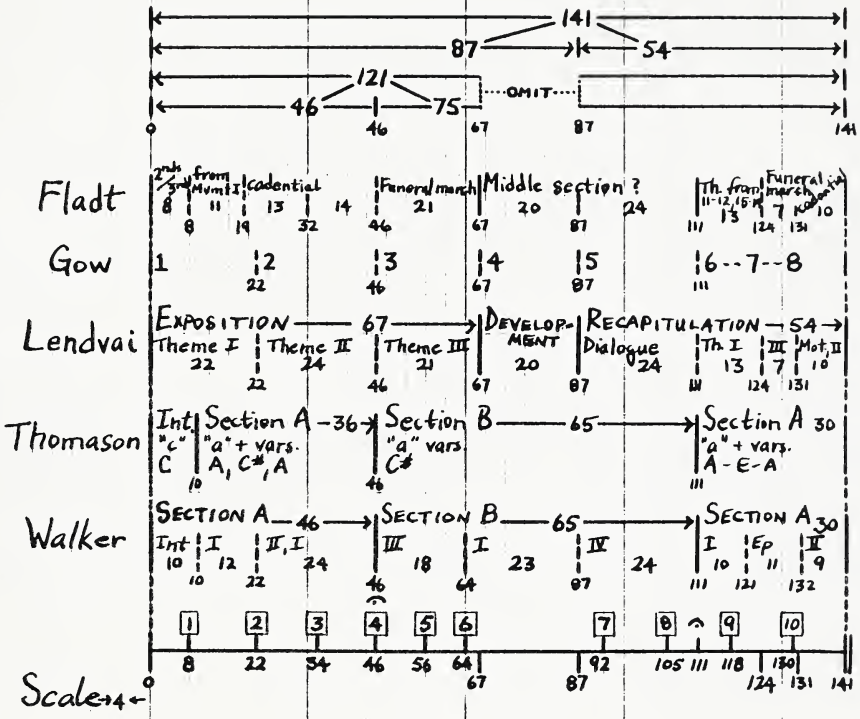

141
87
54
121
46
75
OMIT
0
46
67
87
141
Fladt
2nds/3rds
from Mvmt I
cadential
Funeral march
Middle section ?
Th. from 11-12, 15 K
Funeral march
8
11
13
14
21
20
24
13
7
10
8
19
32
46
67
87
111
124
131
Gow
1
2
3
4
5
6--7--8
22
46
67
87
111
Lendvai
EXPOSITION — 67
DEVELOPMENT
RECAPITULATION — 54
Theme I
Theme II
Theme III
Dialogue
Th. I
III
Mot. II
22
24
21
20
24
13
7
10
22
46
67
87
111
124
131
Thomason
Int.
"c"
C
Section A -36
"a" + vars.
A, C#, A
Section B — 65
"a" vars
C#
Section A 30
"a" + vars.
A-E-A
10
46
111
Walker
SECTION A — 46
SECTION B — 65
SECTION A 30
Int
I
II, I
III
I
IV
I
Ep
II
10
12
24
18
23
24
10
11
9
10
22
46
64
87
111
121
132
1
2
3
4
5
6
7
8
9
10
Scale
0
8
22
34
46
56
64
67
87
92
105
111
118
124
130
131
141

Figure 34. Quartet II, movement 3

Figure 35. Quartet III

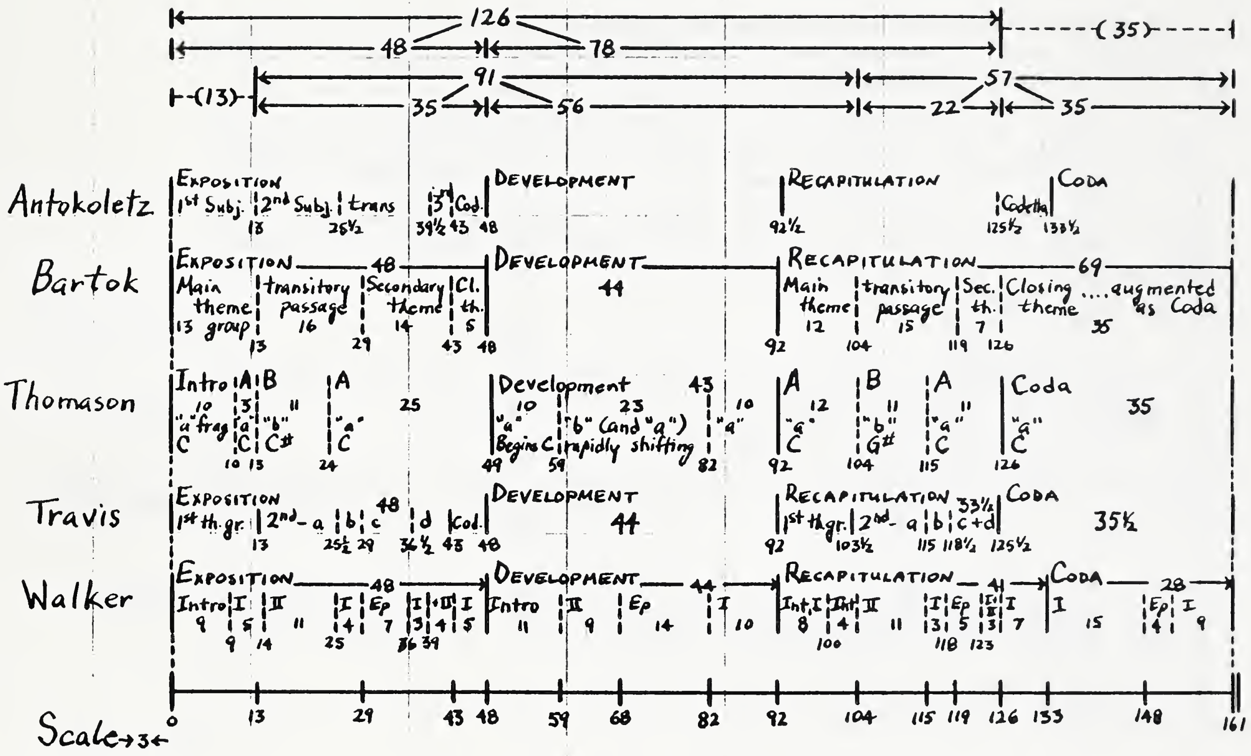

126
48
78
(35)
(13)
91
57
35
56
22
35
Antokoletz
Exposition
1st Subj.
2nd Subj.
trans
3rd
Cod.
Development
Recapitulation
Codetta
Coda
13
25½
39½
43
48
92½
125½
133½
Bartok
Exposition
48
Main theme group
13
transitory passage
16
Secondary theme
14
Cl. th.
5
13
29
43
48
Development
44
Recapitulation
69
Main theme
12
transitory passage
15
Sec. th.
7
Closing theme ... augmented as Coda
35
92
104
119
126
Thomason
Intro
10
"a" frag
C
A
3
"a"
C
B
11
"b"
C#
A
25
"a"
C
10
13
24
Development
10
"a"
Begins C
23
"b" (and "a")
rapidly shifting
43
10
"a"
49
59
82
A
12
"a"
C
B
11
"b"
G#
A
11
"a"
C
Coda
"a"
C
35
92
104
115
126
Travis
Exposition
48
1st th. gr.
2nd - a
b
c
d
Cod.
13
25½
29
36½
43
48
Development
44
Recapitulation
33½
1st th. gr.
2nd - a
b
c+d
Coda
35½
92
103½
115
118½
125½
Walker
Exposition
48
Intro
9
I
5
II
11
I
4
Ep
7
I
3
I+II
4
I
5
9
14
25
36
39
Development
44
Intro
11
II
9
Ep
14
I
10
Recapitulation
41
Int, I
8
Int
4
II
11
I
3
Ep
5
I+II
3
I
7
100
118
123
Coda
28
I
15
Ep
4
I
9
Scale→3←
0
13
29
43
48
59
68
82
92
104
115
119
126
133
148
161

Figure 36. Quartet IV, movement 1

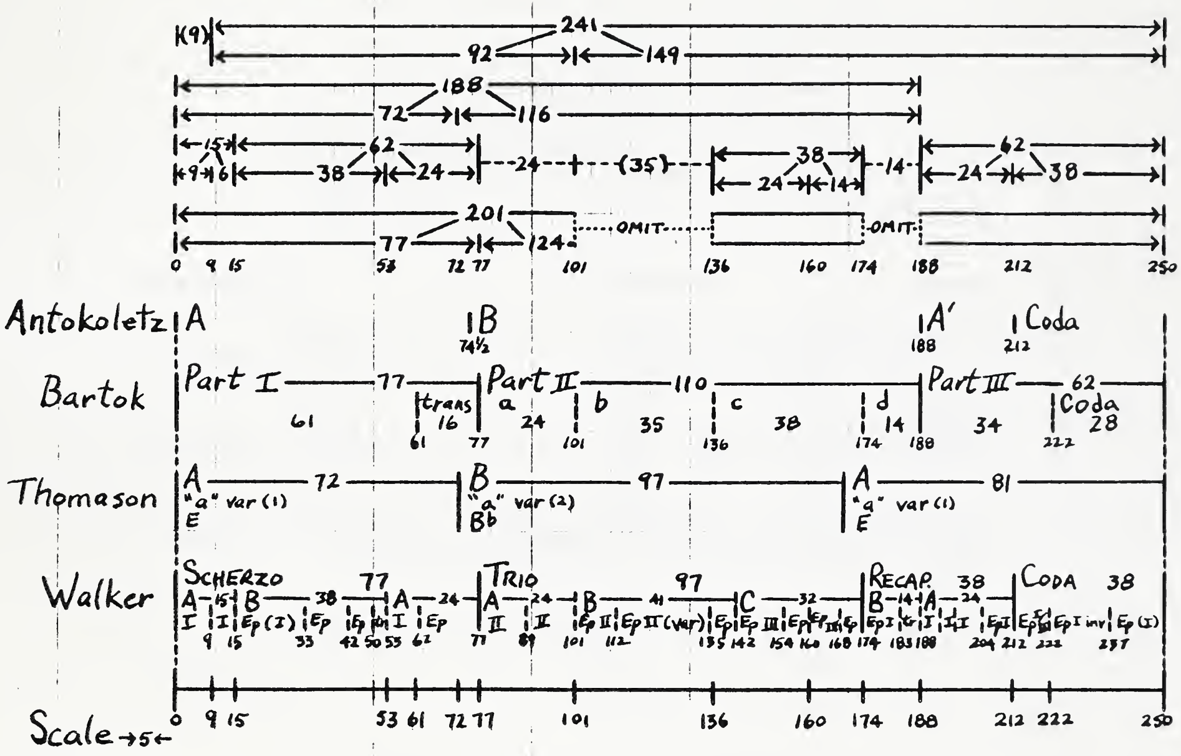

K(9)
241
92
149
188
72
116
15
62
9
6
38
24
24
(35)
38
24
14
14
62
24
38
201
77
124
OMIT
OMIT
0 9 15 53 72 77 101 136 160 174 188 212 250
Antokoletz
A
B
74½
A′
188
Coda
212
Bartok
Part I
77
61
trans
16
61
77
Part II
a
24
b
35
101
110
c
38
136
d
14
174
188
Part III
62
34
Coda
28
222
Thomason
A
"a" var (1)
E
72
B
"a" var (2)
Bb
97
A
"a" var (1)
E
81
Walker
SCHERZO
77
A
15
B
38
I
I
Ep (I)
Ep
Ep
Ep I
A
24
I
Ep
9 15 33 42 50 53 62
TRIO
97
A
24
B
41
II
II
Ep II
Ep II (var)
77 89 101 112
C
32
Ep
Ep III
Ep
Ep III
Ep
135 142 154 160 168
RECAP.
38
B
14
A
24
Ep I
I
I
I I
Ep I
174 183 188 204 212
CODA
38
Ep III
Ep I inv
Ep (I)
222 237
Scale →5←
0 9 15 53 61 72 77 101 136 160 174 188 212 222 250

Figure 37. Quartet IV, movement 2

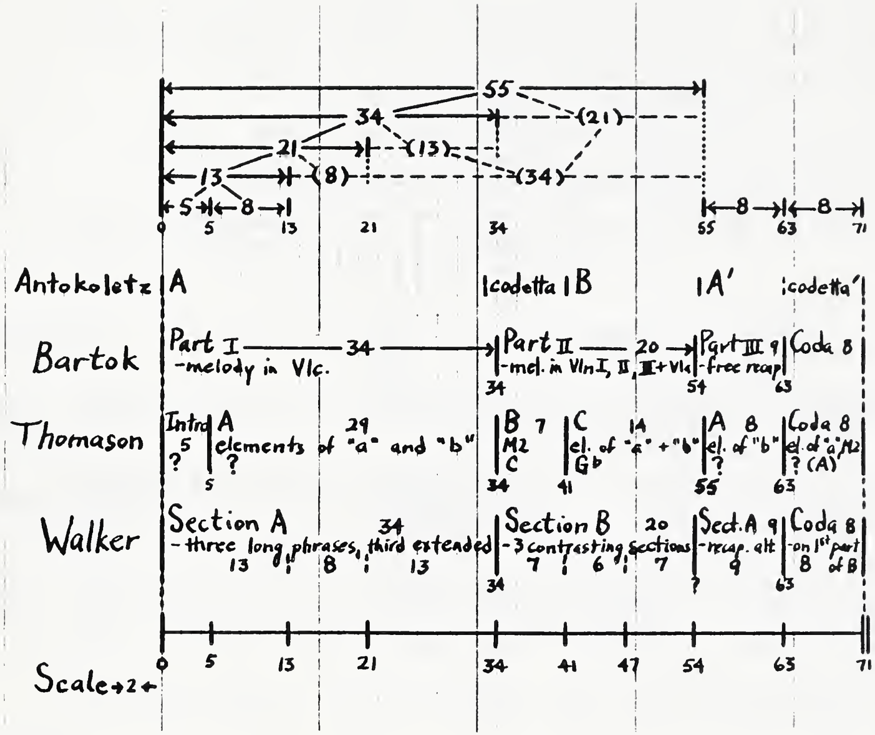

55
34
(21)
21
(13)
13
(8)
(34)
5
8
8
8
0
5
13
21
34
55
63
71
Antokoletz
A
codetta
B
A'
codetta'
Bartok
Part I
34
-melody in Vlc.
Part II
20
-mel. in Vln I, II, III + Vlc
Part III 9
-free recap
Coda 8
34
54
63
Thomason
Intro
5
?
A
elements of "a" and "b"
29
?
5
B 7
M2
C
34
C
14
el. of "a" + "b"
G♭
41
A 8
el. of "b"
?
55
Coda 8
el. of "a" M2
? (A)
63
Walker
Section A
34
-three long phrases, third extended
13
8
13
Section B
20
-3 contrasting sections
7
6
7
34
Sect. A 9
-recap. alt
9
?
Coda 8
-on 1st part of B
8
63
Scale→2←
0
5
13
21
34
41
47
54
63
71

Figure 38. Quartet IV, movement 3

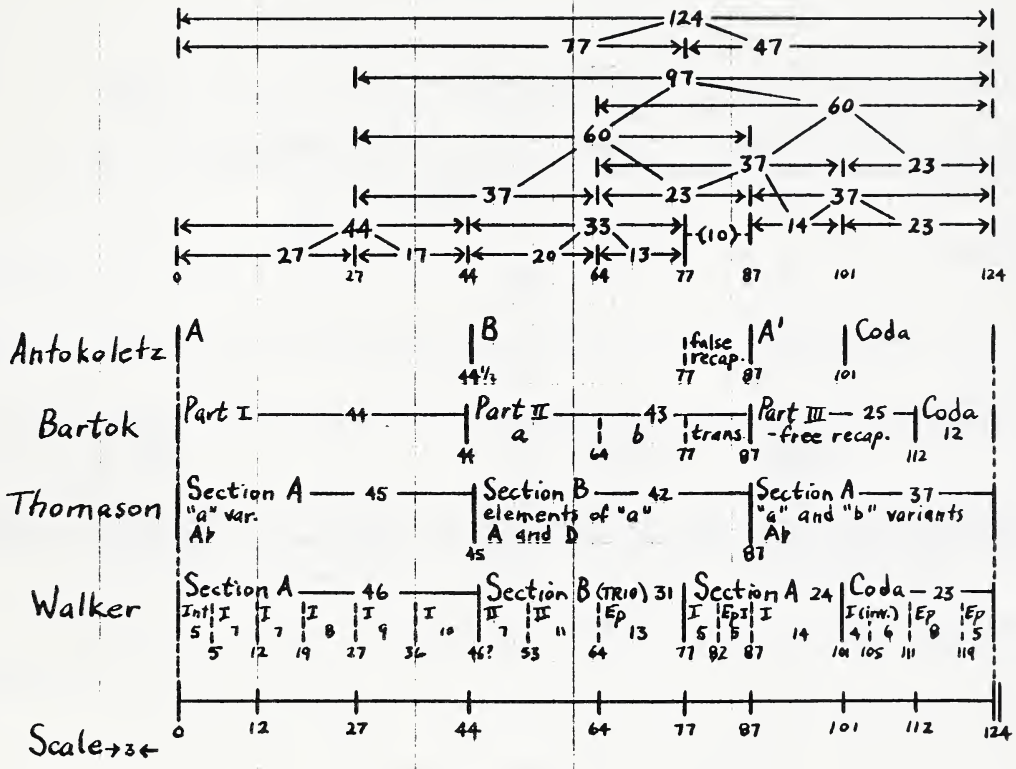

124
77
47
97
60
60
37
23
37
23
37
44
33
(10)
14
23
27
17
20
13
0 27 44 64 77 87 101 124
Antokoletz
A
B
44½
false recap.
77
A'
87
Coda
101
Bartok
Part I — 44
Part II a
44
b
64
43
trans.
77
Part III — 25 -free recap.
87
Coda 12
112
Thomason
Section A — 45
"a" var.
A♭
Section B — 42
elements of "a"
A and D
45
Section A — 37
"a" and "b" variants
A♭
87
Walker
Section A — 46
Int 5
I 7
I 7
I 8
I 9
I 10
5 12 19 27 36
Section B (TRIO) 31
II 7
II 11
Ep 13
46? 53 64
Section A 24
I 5
Ep I 5
I 14
77 82 87
Coda — 23
I (inv.) 4
6
Ep 8
Ep 5
101 105 111 119
Scale →3←
0 12 27 44 64 77 87 101 112 124

Figure 39. Quartet IV, movement 4

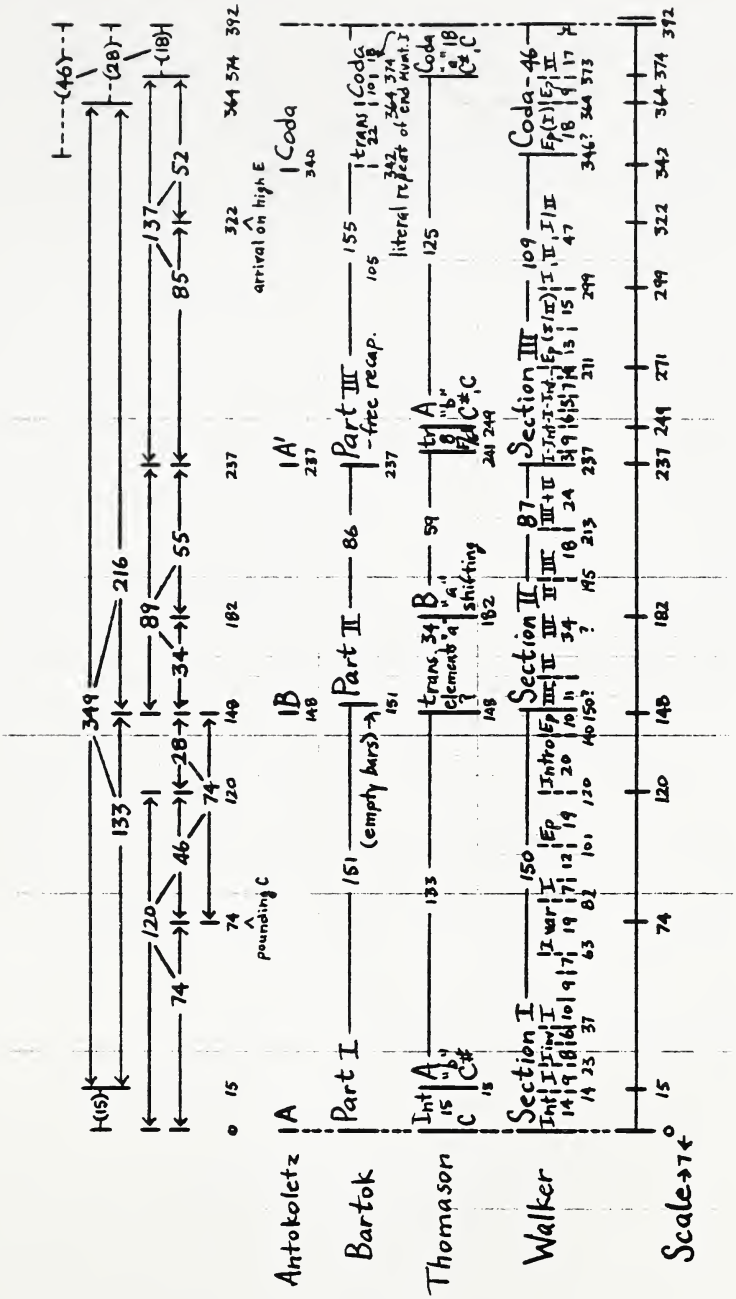

Antokoletz
Bartok
Thomason
Walker
Scale→7←
A
IB
IA'
I Coda
pounding C
arrival on high E
Part I
Part II
Part III
-free recap.
(empty bars)
trans I Coda
literal repeat of end mvt. I
Int
A
B
A
Coda
trans. 34
element "a"
shifting
Section I
Section II
Section III
Coda-46
Intro
Ep
349
133
216
120
89
137
74
46
28
34
55
85
52
(15)
(28)
(18)
(46)
0
15
74
120
148
182
237
271
299
322
342
364
374
392

Figure 40. Quartet IV, movement 5

Figure 41. Quartet V, movement 1 (Allegro)

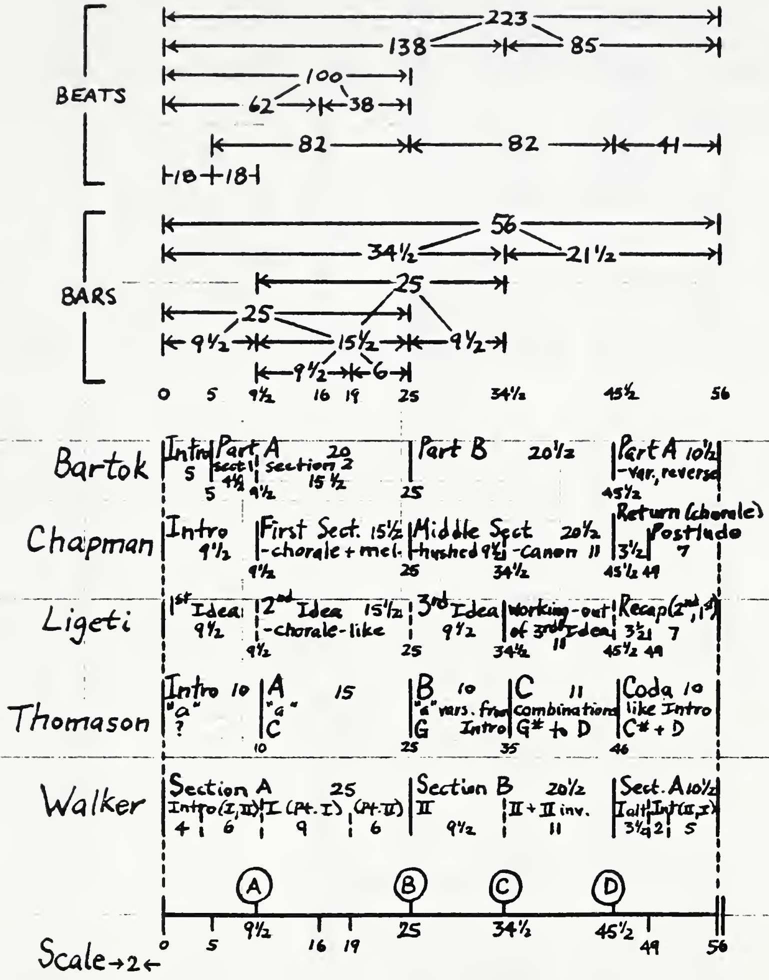

BEATS
223
138
85
100
62
38
82
82
41
18
18
BARS
56
34½
21½
25
25
9½
15½
9½
9½
6
0 5 9½ 16 19 25 34½ 45½ 56
Bartok
Intro 5
Part A 20
sect 1 4½
section 2 15½
5 9½
Part B 20½
25
Part A 10½
-var., reverse
45½
Chapman
Intro 9½
First Sect. 15½
-chorale + mel.
9½
Middle Sect 20½
-hushed 9½
-Canon 11
25 34½
Return (chorale) 3½
Postlude 7
45½ 49
Ligeti
1st Idea 9½
2nd Idea 15½
-chorale-like
9½
3rd Idea 9½
25
working-out of 3rd Idea 11
34½
Recap (2nd, 1st) 3½ 7
45½ 49
Thomason
Intro 10
"a"
?
A 15
"a"
C
10
B 10
"a" vars. from Intro
G
25
C 11
combinations
G# to D
35
Coda 10
like Intro
C# + D
46
Walker
Section A 25
Intro (I, II) 4 6
I (Pt. I) 9
(Pt. II) 6
Section B 20½
II 9½
II + II inv. 11
Sect. A 10½
I alt. 3½
Int (II, I) 2 5
A
B
C
D
0 5 9½ 16 19 25 34½ 45½ 49 56
Scale→2←

Figure 42. Quartet V, movement 2
(Adagio molto)

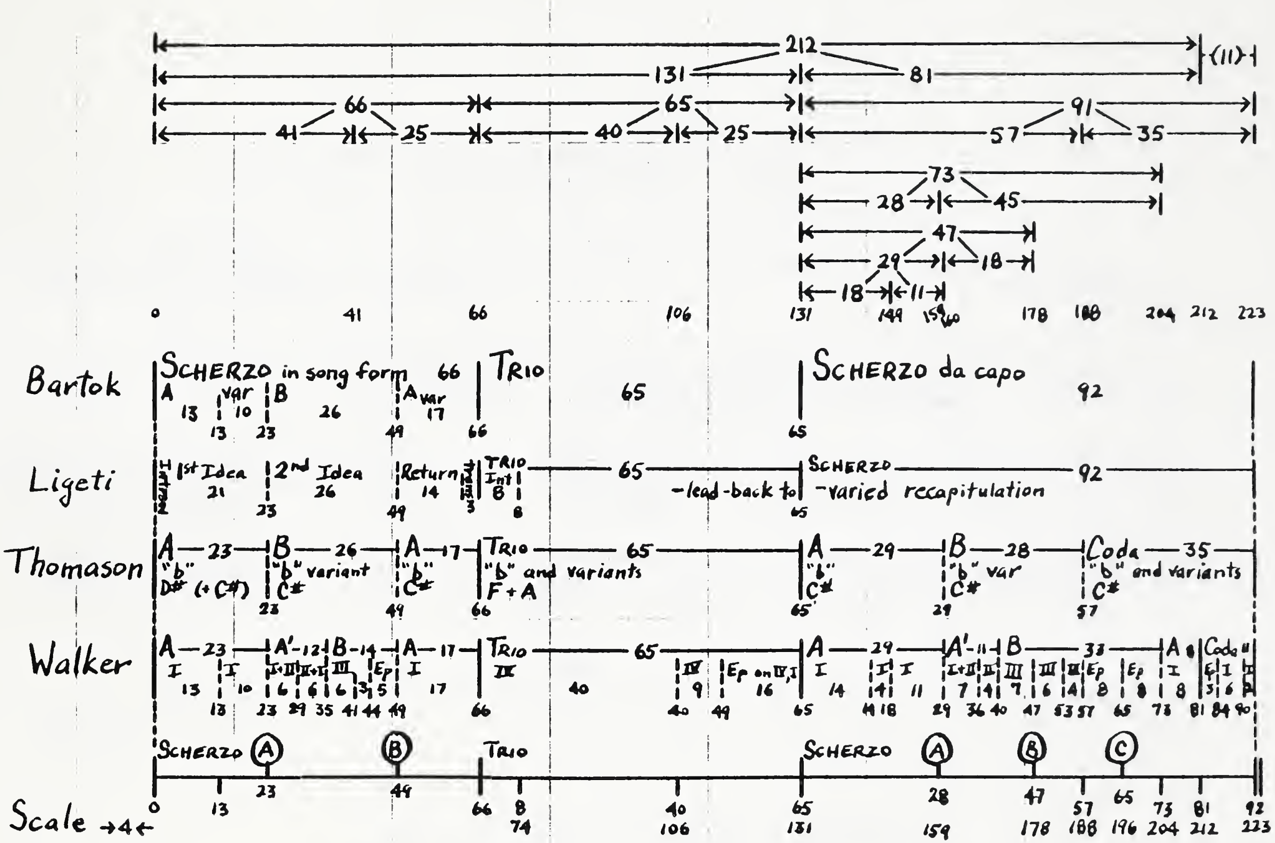

212
(11)
131
81
66
65
91
41
25
40
25
57
35
73
28
45
47
29
18
18
11
0
41
66
106
131
149
159
160
178
188
204
212
223
Bartok
SCHERZO in song form
A
var
B
A var
TRIO
SCHERZO da capo
Ligeti
1st Idea
2nd Idea
Return
TRIO
Int
B
-lead-back to
SCHERZO
-varied recapitulation
Thomason
A
B
Coda
"b"
D# (+C#)
"b" variant
C#
"b" and variants
F+A
"b" var
Walker
A'
Ep
Ep on IV, I
Coda
Scale →4←
SCHERZO
Ⓐ
Ⓑ
TRIO
Ⓒ

Figure 43. Quartet V, movement 3 (Scherzo)

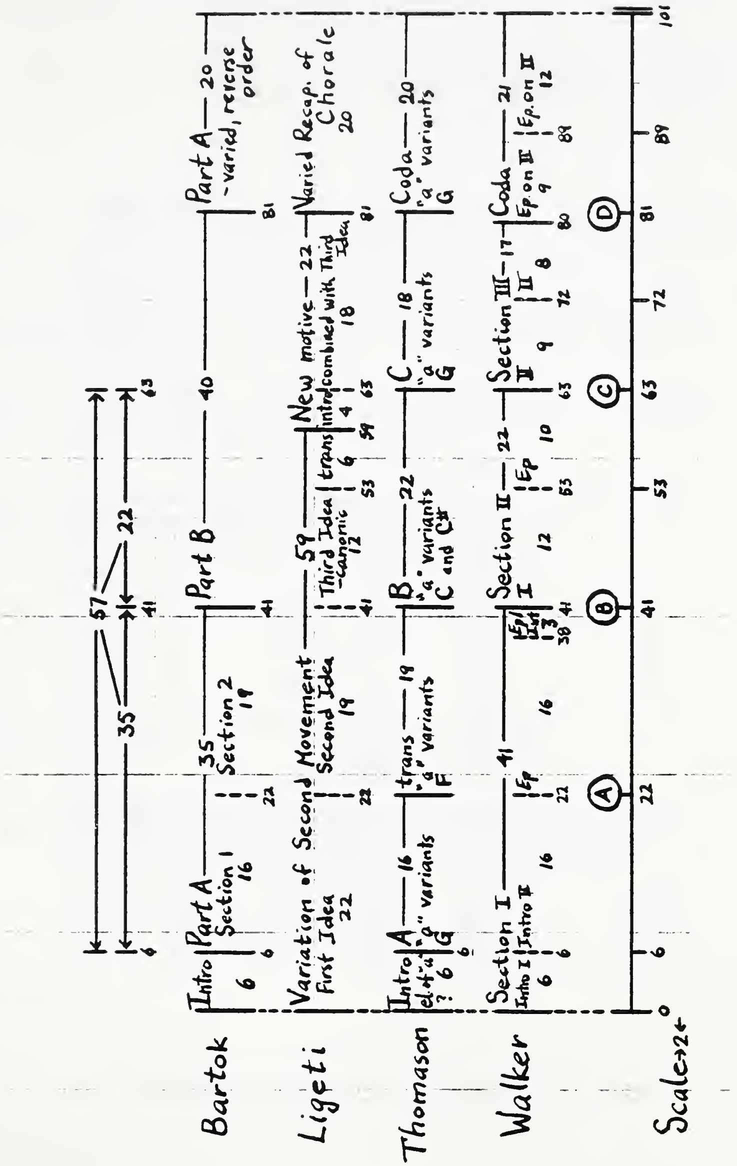

Bartok
Intro
6
Part A
Section 1
16
35
Section 2
19
Part B
40
Part A
20
-varied, reverse order
Ligeti
Variation of Second Movement
First Idea
22
Second Idea
19
59
Third Idea
-canonic
12
trans
6
New motive
intro
4
combined with Third Idea
18
22
Varied Recap. of Chorale
20
Thomason
Intro
el. of "a"
? 6
A
"a" variants
G
16
trans
"a" variants
F
19
B
"a" variants
C and C#
22
C
"a" variants
G
18
Coda
"a" variants
G
20
Walker
Section I
Intro I
6
Intro II
16
Ep
41
16
Ep/Intr
3
Section II
I
12
Ep
10
22
Section III
II
9
III
8
17
Coda
Ep. on II
9
Ep. on II
12
21
A
B
C
D
35
57
22
Scale→2←
0
6
21
22
38
41
53
59
63
72
80
81
89
101

Figure 44. Quartet V, movement 4 (Andante)

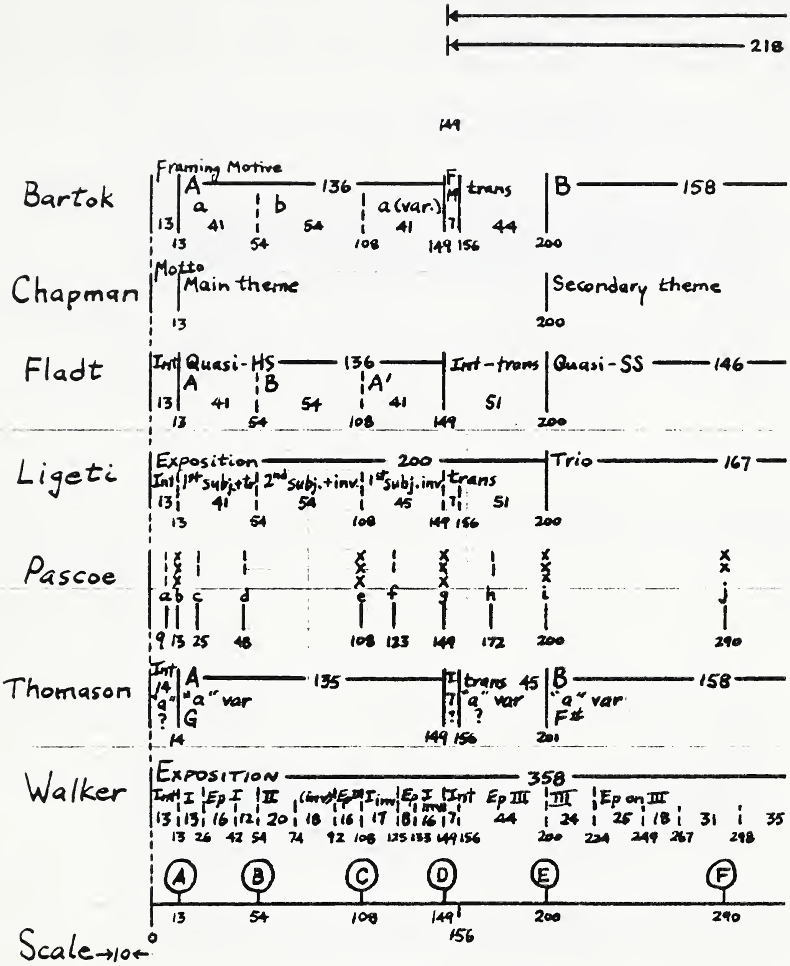
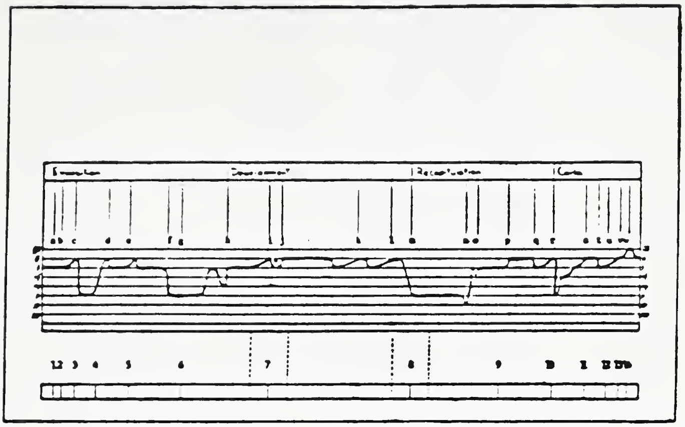
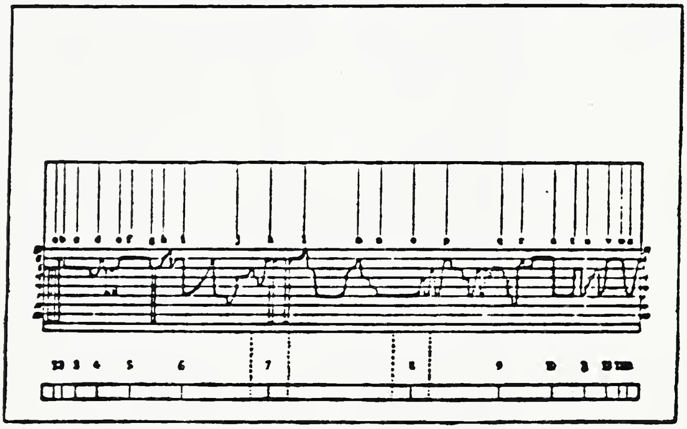

218
149
Bartok
Framing Motive
A
136
a
b
a (var.)
F
M
trans
B
158
13
41
54
41
7
44
13
54
108
149
156
200
Chapman
Motto
Main theme
Secondary theme
13
200
Fladt
Int
Quasi-HS
136
A
B
A'
Int-trans
Quasi-SS
146
13
41
54
41
51
13
54
108
149
200
Ligeti
Exposition
200
Trio
167
Int
1st subj.+tr.
2nd subj.+inv.
1st subj. inv.
trans
13
41
54
45
7
51
13
54
108
149
156
200
Pascoe
a
b
c
d
e
f
g
h
i
j
9
13
25
48
108
123
149
172
200
290
Thomason
Int
14
"a"
?
A
"a" var
G
135
I
7
?
trans
45
"a" var
?
B
"a" var
F#
158
14
149
156
201
Walker
Exposition
358
Int
I
Ep I
II
(inv) Ep II
I inv
Ep I inv
Int
Ep III
III
Ep on III
13
13
16
12
20
18
16
17
18
16
7
44
24
25
18
31
35
13
26
42
54
74
92
108
125
133
149
156
200
224
249
267
298
A
B
C
D
E
F
13
54
108
149
156
200
290
Scale →10←
0

Figure 45. Quartet V, movement 5 (Finale)



a. Quartet V, movement 1



b. Quartet V, Finale

Figure 46. Pascoe's graphs of form structure in relation to golden modules

Figure 47. Quartet VI, movement 1

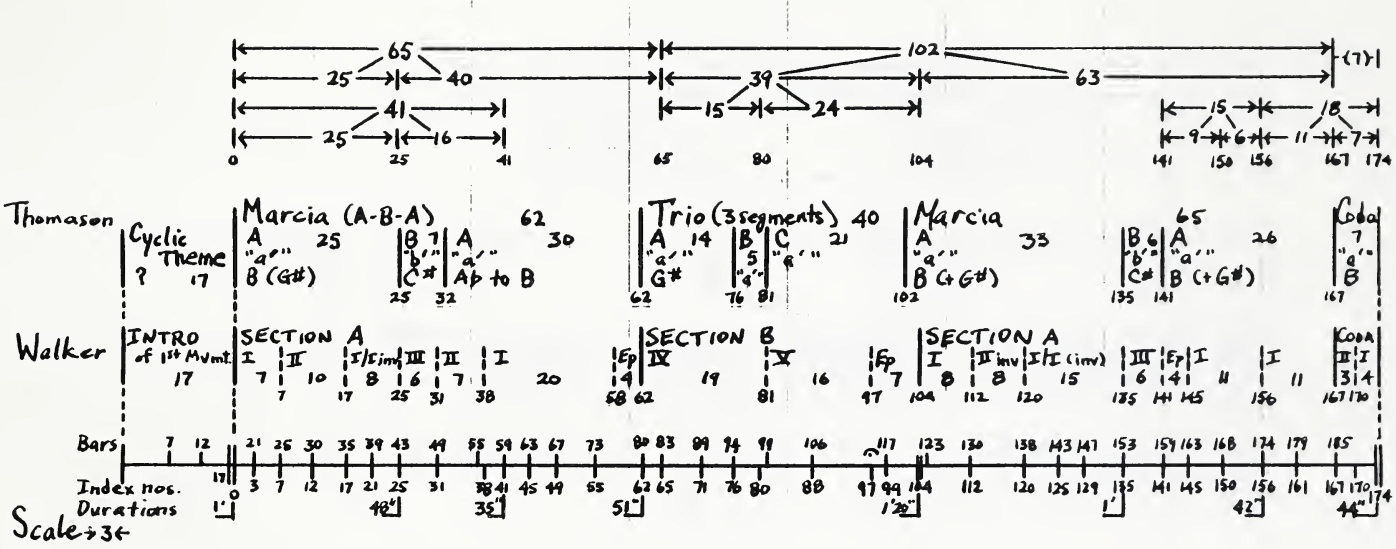

Thomason
Cyclic Theme ? 17
Marcia (A-B-A) 62
A 25 "a'" B (G#)
B 7 "b'" C#
A 30 "a'" Ab to B
Trio (3 segments) 40
A 14 "a'" G#
B 5 "a'"
C 21 "a'"
Marcia 65
A 33 "a'" B (+G#)
B 6 "b'" C#
A 26 "a'" B (+G#)
Coda 7 "a'" B
Walker
INTRO of 1st Mvmt. 17
SECTION A
I 7
II 10
I/I inv 8
III 6
II 7
I 20
Ep 4
SECTION B
IV 19
V 16
Ep 7
SECTION A
I 8
II inv 8
I/I (inv) 15
III 6
Ep 4
I 11
I 11
CODA
II 3
I 4
Bars
Index nos.
Durations
Scale→3←
65
25
40
41
25
16
102
39
15
24
63
(7)
15
18
9
6
11
7
1'
48"
35"
51"
1'20"
1'
42"
44"

Figure 48. Quartet VI, movement 2

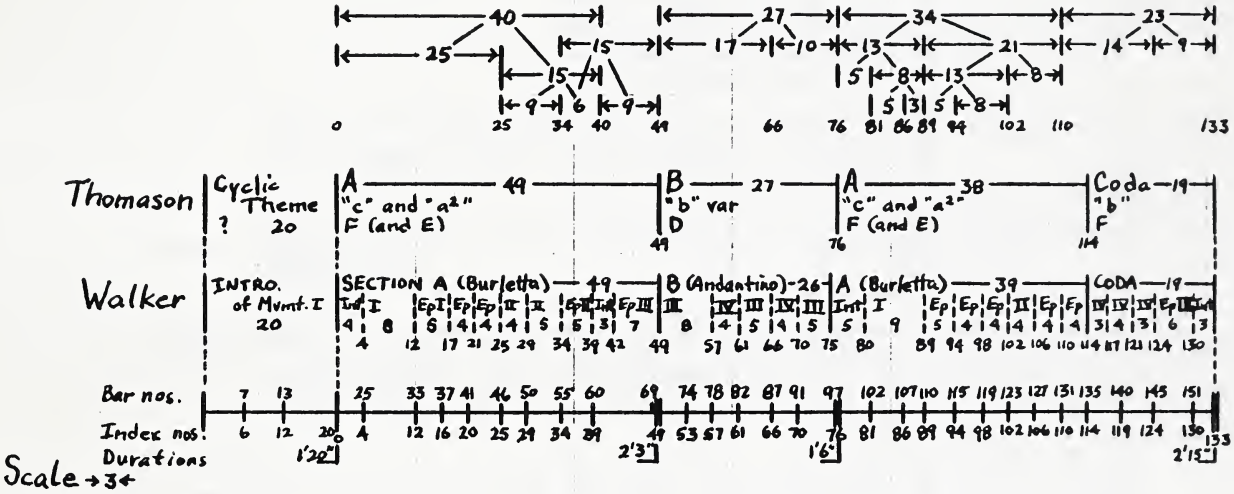

Thomason
Cyclic Theme
? 20
A "c" and "a²" F (and E) 49
B "b" var D 27
A "c" and "a²" F (and E) 38
Coda "b" F 19
Walker
INTRO. of Mvmt. I 20
SECTION A (Burletta) 49
B (Andantino) 26
A (Burletta) 39
CODA 19
Bar nos.
Index nos.
Durations
1'20"
2'3"
1'6"
2'15"
Scale → 3 ←
40 25 15 15 9 6 9
27 17 10
34 13 21 5 8 13 8 5 3 5 8
23 14 9
0 25 34 40 49 66 76 81 86 89 94 102 110 133

Figure 49. Quartet VI, movement 3

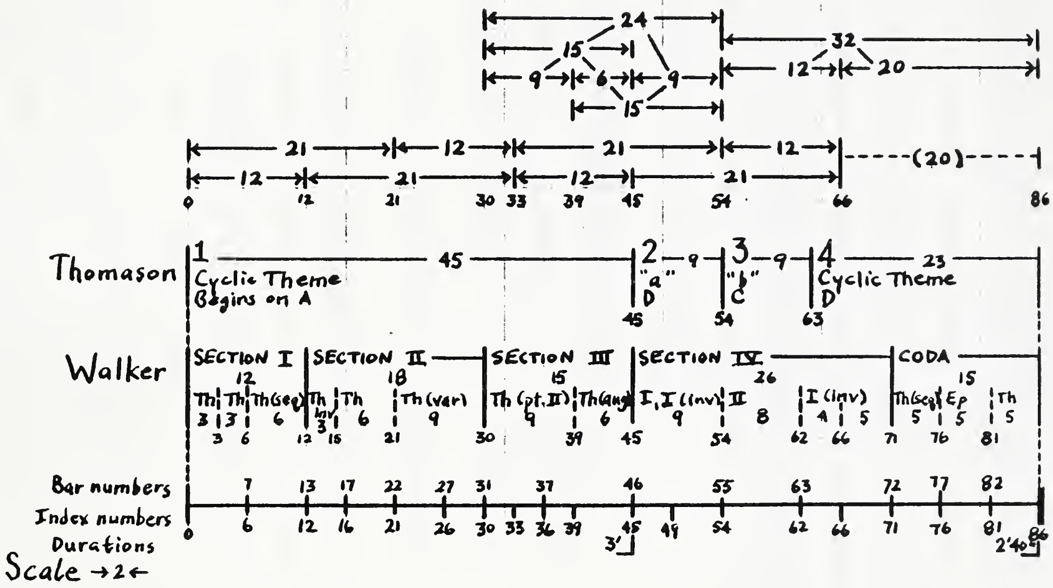

Thomason
Walker
Bar numbers
Index numbers
Durations
Scale →2←
24
32
15
12
20
9
6
9
15
21
12
21
12
(20)
12
21
12
21
0 12 21 30 33 39 45 54 66 86
1 Cyclic Theme Begins on A
45
2 "a" D
3 "b" C
4 Cyclic Theme D
9
9
23
45 54 63
SECTION I 12
SECTION II 18
SECTION III 15
SECTION IV 26
CODA 15
Th 3 Th 3 Th(seq) 6 Th Inv 3 Th 6 Th (var) 9
Th (pt. II) 9 Th(aug) 6
I, I (inv) 9 II 8 I (inv) 4 5
Th(seq) 5 Ep 5 Th 5
3 6 12 15 21 30 39 45 54 62 66 71 76 81
7 13 17 22 27 31 37 46 55 63 72 77 82
0 6 12 16 21 26 30 33 36 39 45 49 54 62 66 71 76 81 86
3'
2'40"

Figure 50. Quartet VI, movement 4

CHAPTER III

SYSTEMS OF GOLDEN SECTION

The simplest application of golden section in the formal partitioning of a musical movement would be the placement of a highly significant musical event at the point of golden section of the entire movement. In terms of the system of modules described in Chapter I (p. 10), this would divide a whole length into two modules, 1 and 2, either positively (larger module first) or negatively (smaller module first; Figure 51).

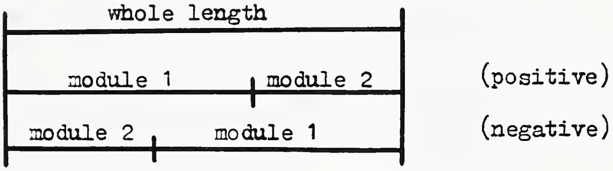


Figure 51. First level of golden section

This type of golden section may be found in many of Bartok's string quartet movements, beginning with the first movement of the First Quartet. Analysts agree that this movement is in ternary form (*A* - *B* - *A'*), and Walker and Fladt see the *B* section as subdivided in three parts (*a* - *b* - *a'*) as well. The movement is seventy-one bars in length; the central subdivision (*b*) of the central section (*B*) begins after forty-four bars, the point of positive golden section of the movement (Figure 52).

Without referring to the golden symmetry of the movement, Fladt calls this point the "center of the movement," pointing out that the cello line in bar 45 attains greater significance in the final movement, where it serves as the theme of the Adagio section.[1]

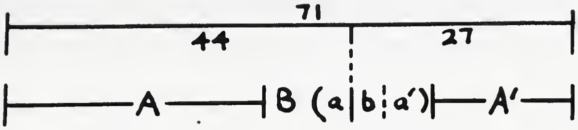


Figure 52. Golden section of Quartet I, movement 1

Similarly, the first movement of the Second Quartet (180 bars in length) is subdivided by golden section—negatively this time—after sixty-nine bars, the point at which the development (or B section of a ternary—Thomason) begins (Figure 53).

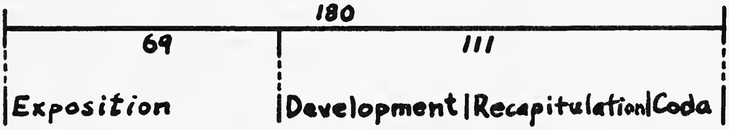


Figure 53. Golden section of Quartet II, movement 2

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[1] Hartmut Fladt, Zur Problematic traditioneller Formtypen . . . (München: Musikverlag Katzbichler, 1974), p. 88.

Other examples of movements displaying the first level of golden section (into modules 1 and 2) over their entire length are the third movement of the Second Quartet (141 bars divided positively into modules of 87 and 54 bars), the second movement of the Fifth Quartet (56 bars divided positively into modules of $34\frac{1}{2}$ and $21\frac{1}{2}$ bars[2] at letter C), and the first movement of the Sixth Quartet (377 bars divided negatively into modules of 144 and 233 bars, these being Fibonacci numbers).[3]

---

[2] Regarding the use of half-bar units, it is interesting to note that half-bar lines are used in bars 10 and 46, with the letters A and D placed directly over the center of the bar. In bar 35, no half-bar line is used, but rests clearly divide the bar into halves and the letter C is placed above the center of the bar. In contrast, letter B aligns with the bar line beginning bar 26. This would seem to indicate that Bartok was thinking in half-bar units for this particular movement.

[3] See Chapter V, p. 123, for further discussion of this example. In addition, the third movement of the First Quartet divides positively at number 27 (380/235/145), at which point the E pedal which begins the work returns; however, most analysts consider this a retransition, the recapitulation proper beginning at number 28 (after 249 bars). Similarly, the fourth movement of the Fourth Quartet divides positively after 77 bars (124/77/47), but at a point which Bartok labels a transition; Elliott Antokoletz, "Principles of Pitch Organization in Bartok's Fourth String Quartet" (Ph.D. dissertation, City University of New York, 1975), calls it a false recapitulation, while Walker sees it as the point of return of section A of a ternary. The second movement of the Second Quartet may divide negatively at number 19 (577/220/357), but most analysts see the middle section beginning just prior to (after 215 bars) or after (at number 20, 233 bars) this point.

The first level of golden section may also be seen to operate in several other movements if certain sections, such as codas, are excluded. For instance, the first movement of the Fourth Quartet divides negatively by golden section at the beginning of the development (126/48/78) if the coda (bars 127-161) is excluded. The third movement of the same quartet divides according to positive Fibonaccian proportioning (55/34/21) if the last sixteen bars—the free recapitulation and coda of eight bars each—are omitted.[4] The final movement of this quartet is sectioned negatively at the beginning of Part II (349/133/216) omitting the fifteen-bar introduction and twenty-eight-bar coda.[5] Similarly, the Scherzo of the Fifth Quartet is positively proportioned at the Scherzo da capo (212/131/81), excluding an eleven-bar coda.

---

[4] See further discussion of this movement, p. 132.

[5] Antokoletz claims the entire movement, including the introduction and coda, is divided by negative golden section, since the total of 392 bars times 0.382 equals 149.7444 and the B section begins at bar 149; however, 149.7444 rounds upward to 150, and if a section begins at bar 149, then only 148 bars have elapsed and 244 bars remain.

In the discussion of modules of golden proportion in Chapter I (pp. 9-13), it was noted that the process of golden section is:

a) hierarchical—it may operate continuously at various levels, yielding smaller and smaller modules; and

b) symmetrical--it may divide a length either positively or negatively, the one result being a reflection of the other.

The use of these properties for more systematic proportioning of musical space may also be seen in the Bartok Quartets, again beginning with the opening movement of the First Quartet. While it was noted that the overall movement divides at the "center of the movement" (71/44/27), a separate hierarchical system of golden proportion operates over the first two sections of the ternary design. The fifty-two-bar length is divided according to four golden modules: the first section is thirty-two bars in length; the second, twenty. The first is subdivided into lengths of twenty and twelve;[6] the second, twelve and eight (52/32/20/12/8; Figure 54).

---

[6] Thomason identifies two basic motives as being significant in this movement, and divides the first section after bars 10 and 24 on the basis of her motivic analysis. It seems to the present writer that bars 11 and 12 are transitional in nature, the real emphasis on the second motive beginning in bar 13 with the exposed and accented C# to F# figure in the first violin.

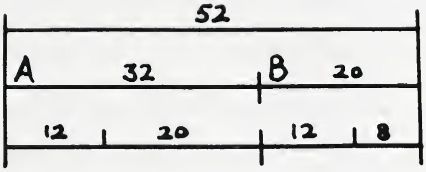


Figure 54. Hierarchical division of Quartet I, movement 1, A and B

Similarly, the Introduzione to the third movement of the same quartet, a length of thirty-three bars, divides in modules of twenty, thirteen, eight and five, with a central portion of seven bars. (A Fibonaccian length of thirty-four would have provided a better whole-number approximation--34/21/13/8/5.) Because the irregular length is central, the overall structure of the Introduzione is symmetrical at both the lower level (8-5-7-5-8) and intermediate level (13-7-13). At the upper level, the division could be seen as either positive (20-13) or negative (13-20), depending on whether the central Meno vivo cello solo or the climactic Molto Adagio was felt to have greater emphasis (Figure 55).

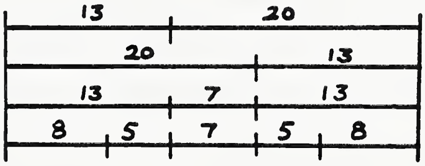
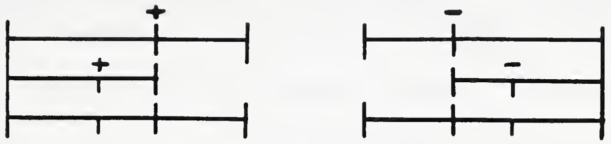


Figure 55. Symmetrical division of Quartet I, Introduzione to movement 3

This situation may be described as a positively sectioned length in which the larger module is also positively sectioned (Figure 56a), or as a negatively sectioned length in which the larger module is also negatively sectioned (Figure 56b).



a. Positive b. Negative

Figure 56. Subsequent sections of similar polarity

In either case, the result is a length divided into three sections of two different golden modules, with the smaller module central; this might be referred to as a contained system, as it constitutes a self-contained system in which each module is in golden relationship to adjacent modules. The order and polarity of the section and sub-section which yielded the system are indeterminate due to the symmetry of the result; that is, whether the smaller, central module is thought of as the end of a larger, opening module or as the beginning of a larger, closing module is either immaterial or dependent on the musical relationships between the modules rather than their proportional relationships (Figure 57).

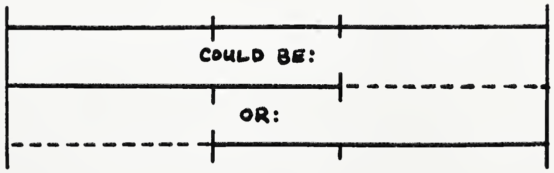


Figure 57. Indeterminacy of polarity in a contained system

Examples of contained systems may be found in the third movement of the First Quartet between numbers 19 and 36 (162/100/62/38); in the fourth movement of the Fourth Quartet, bars 28 to 124 (97/60/37/23/14); in the third movement of the Fifth Quartet, from the Scherzo da capo to letter B (47/29/18/11); in both the first movement, bars 47 to 80, and the third movement, bars 97 to 130, of the Sixth Quartet (of identical length, 34/21/13/8/5/3, Fibonacci systems);[7] and in both the third movement, bars 47 to 69, and the fourth movement, bars 31 to 54, of the Sixth Quartet (again of identical length, 24/15/9/6).

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[7] See discussion in Chapter V, pp. 125-127, 133.

Another type of system, opposite to the contained system, is that in which a length is divided into three sections of two different golden modules but, in contrast to the contained system, having the larger module central. In this case, neither of the dividing points subdivides the entire length by golden section; rather, it is as though two equal lengths divided according to golden section—one positively, the other negatively—have been joined or interlocked, sharing a common larger section. This might be referred to as an interlocking system (Figure 58).

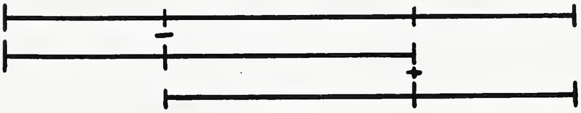

Figure 58. Interlocking system

An example of an interlocking system may be found in the second movement of the Fifth Quartet, from the beginning to letter C (the golden section of the overall movement). The passage from the beginning to letter B is twenty-five bars in length, divided negatively as $9\frac{1}{2}$-$15\frac{1}{2}$; the passage from letter A to letter C is similarly twenty-five bars long, divided positively as $15\frac{1}{2}$-$9\frac{1}{2}$. The shared central portion is subdivided positively as $9\frac{1}{2}$-6 after bar 19. Thus, while all of these sections are in golden relationship to each other (as part of the series $25/15\frac{1}{2}/9\frac{1}{2}/6$), none is in golden relationship to the overall length of $34\frac{1}{2}$ bars, nor to the remainder of the movement (letter C to the end), $21\frac{1}{2}$ bars.

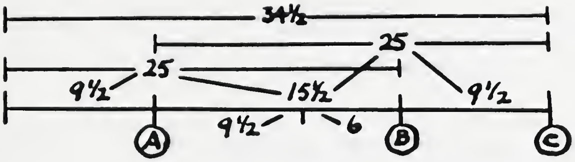

Figure 59. Interlocking system
in Quartet V, movement 2

A combination of a contained and interlocking system is found in the third movement of the First Quartet, between numbers 23 and 38. A contained system forms the central portion of this passage, between number 27 and bar 319--the former being the point of golden section of the entire movement, the latter marking the recurrence of the Adagio episode (as László Somfai[8] calls this secondary thematic material). On either side of this contained system is an interlocking system, subdividing the whole 148-bar passage cited as 32-32-20-32-32 at the lowest level (Figure 60).

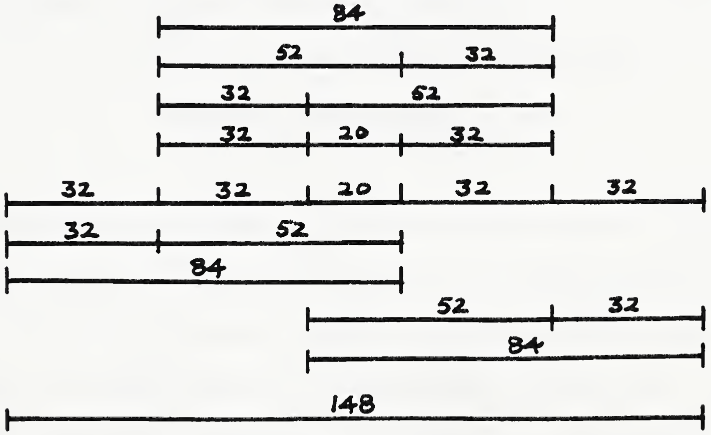

Figure 60. Contained and interlocking systems
in Quartet I, movement 3

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[8] " 'Per finire'; Some Aspects of the Finale in Bartok's Cyclic Form," Studia Musicologica, XI ("Bence Szabolcsi/Septuagenario," 1969), 391-408.

Another type of system is found at the opening of the same movement (Quartet I, movement 3). The series 105/65/40 is used, but successive uses of the series are linearly displaced, forming what might be termed an overlapping system: the first 105 bars (to number 12) divide positively as 65-40 at number 8; the 105-bar passage between numbers 2 and 15 divides similarly at number 11 (Figure 61). The displacement is by twenty-eight bars, a number which has no relationship to the series 105/65/40; however, when combined with the following 65 bars, it forms a 93-bar passage which divides by golden section at number 7 into subsections of 57 and 35 bars.

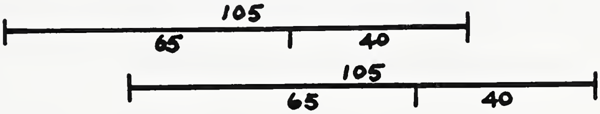

Figure 61. Overlapping systems in Quartet I, movement 3

Furthermore, the second of the overlapping systems is linearly extended. Following the initial section of 65 bars and the second of 40 bars are third and fourth sections of 25 and 15 bars respectively, extending the golden series as 105/65/40/25/15 (Figure 62). This extension subdivides the movement to number 19, at which point the contained system between numbers 19 and 36 and the 148-bar contained and interlocking systems subdivide the remainder of the movement to number 38.

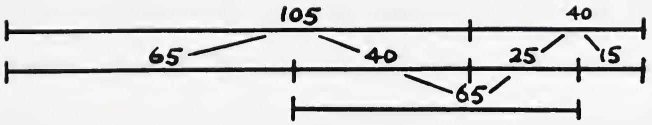

Figure 62. Linearly-extended system
in Quartet I, movement 3

Another linearly-extended system is found in the third movement of the Fourth Quartet. A five-bar introduction followed by an eight-bar phrase to a cadence on E forms a thirteen-bar section; this is extended by another eight-bar phrase to a cadence on C after twenty-one bars, and concluded by a thirteen-bar phrase to another cadence on C, the first section totalling thirty-four bars.[9] A twenty-one-bar second section leads to a free recapitulation and coda of eight bars each after fifty-five bars (Figure 63). Each of these numbers belongs to the Fibonacci series, suggesting that Bartok was here consciously contrasting--or perhaps integrating--the golden section series in its best whole-number approximation and the geometric series in its most familiar Classic form (8 + 8 = 16).

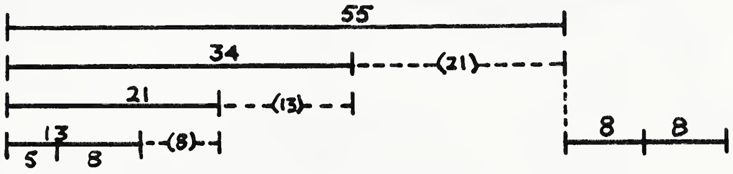

Figure 63. Linearly-extended system
in Quartet IV, movement 3

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[9] This 34-bar section could also be viewed as a contained system of 13-, 8-, and 13-bar modules.

The fifth movement of the same quartet displays yet another linearly-extended system. As Colin Mason points out:

> In the finale, just as at the beginning of the recapitulation of the first movement Bartok had strongly asserted his C, so here he begins with a pounding chord on C . . . sustained for 42 bars, then reappearing at bar 75 for another 15 bars . . . " 10

Mason also stresses the significance of the tritone D-A♭ in the Fourth Quartet, and says regarding the last movement that "although in this movement as a whole there is less stress on this particular tonal relationship it is very prominent in bars 121-145."[11]

As the first section of this movement ends in bar 148 (with a return to C), followed by over three measures of silence, the proportioning of this section would appear to be 74 bars - 46 bars - 28 bars. These lengths happen to be in golden proportion to one another, modules of the golden series 120/74/46/28/18. As a result of their linear extension, the overall passage of 148 bars is divided not by golden section but into two equal parts of 74 bars (Figure 64).

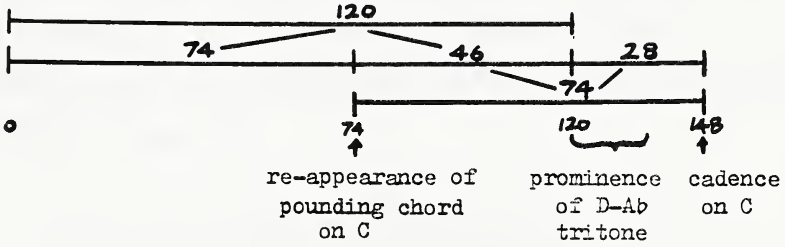
120
74
46
28
74
0
74
120
148
re-appearance of pounding chord on C
prominence of D-A♭ tritone
cadence on C

Figure 64. Linearly-extended system in Quartet IV, movement 5

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10 "An Essay in Analysis," Music Review, XVIII (1957), 200.

[11] Ibid., p. 194.

It is interesting to note that while the analysts disagree as to the extent of the coda of this movement, three of them designate lengths which are modules of this same golden series: Walker sees a 46-bar coda beginning in bar 347; Bartok identifies a 28-bar coda beginning at bar 365, "the second half of which (from 374, _Meno mosso_) is an almost literal repetition of the close of Movement I";[12] Thomason names the latter 18-bar portion as the coda. Thus the ending of this movement could be seen as a separated continuation of the series of golden modules with which it begins (Figure 65).

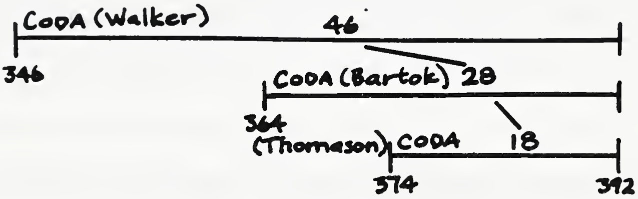

Figure 65. Continuation of the linearly-extended system in Quartet IV, movement 5

A final example of a linearly-extended system is the opening 49-bar _Burletta_ in the third movement of the Sixth Quartet, in which the last nine bars are a modular extension of the main 40-bar system (40/25/15/9/6; Figure 66).

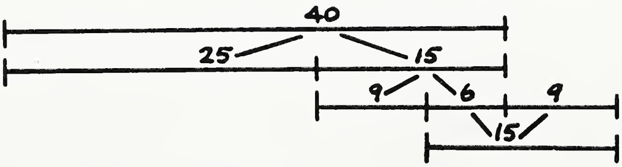

Figure 66. Linearly-extended system in Quartet VI, movement 3, _Burletta_

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[12] "Structure of the Fourth String Quartet," _Bela Bartok Essays_, p. 413.

The first movement of the Fifth Quartet displays yet another intricate set of golden proportions in what might be termed a diagonally-related system. The 34-bar section between letters F and I is sectioned into 21- and 13-bar golden modules at letter H; this point is also the start of a 55-bar passage to letter K. The numbers 13, 21, 34, and 55 are the seventh through tenth terms of the Fibonacci series. The diagonal relationship is of interest in that the 55-bar module begins at the point of golden section of the 34-bar module (letter H) rather than at its beginning (letter F); thus, rather than forming a contained system with the adjacent 21 bars, it encompasses 42 bars which are otherwise unrelated to the proportional system (Figure 67).

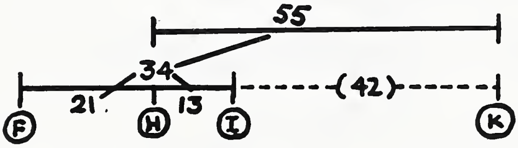

Figure 67. Diagonally-related system
in Quartet V, movement 1

Furthermore, the $44\frac{1}{2}$-bar passage from letter G to J is sectioned into golden modules of $27\frac{1}{2}$ and 17 bars at letter I. The next-larger module of this series would be 72 bars, and that is the length of the passage from letter H to the end of the movement. (Letter H is also the origin of the 55-bar passage mentioned above.) Interestingly, the number of proportionally unrelated bars encompassed by this diagonal relationship is again 42. Moreover, the section between letters E and I (also the terminal point of the 55-bar passage) is 116 bars in length, extending to one higher order the golden series $17/27\frac{1}{2}/44\frac{1}{2}/72/116$. It may be significant that the number of bars not covered by the largest module (i.e., letter K to the end) is equal to the smallest module used—17 (Figure 68).[13]

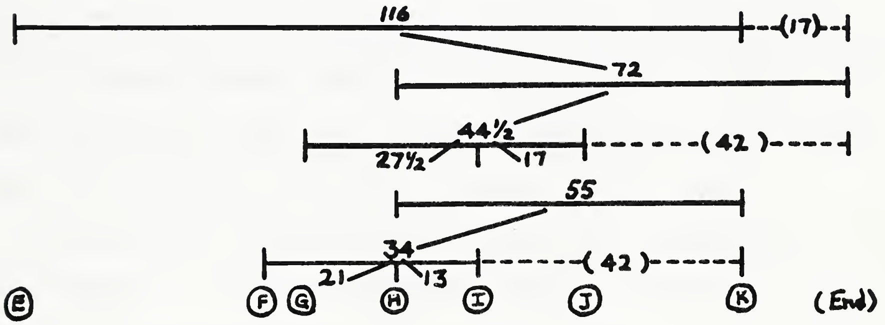

Figure 68. Double diagonally-related system in Quartet V, movement 1

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[13] See also further discussion of this movement in Chapter V. p. 127.

A system such as this is in stark contrast to the neatly hierarchical system of modules diagrammed in Chapter I (Figures 10 and 11, pp. 9-10). It is difficult to conceive of this complex, diagonally-related system as ever being perceived, or indeed, as having been meant to be perceived. However, a number of factors suggest that it was in fact intentionally designed by the composer:

1. All modules identified begin and end at one of the rehearsal letters specified by Bartok, and all the rehearsal letters from E to the end are included in the system.

2. One of the diagonally-related systems consists of modules having Fibonacci numbers of bars, and the origin and termination of the largest Fibonacci module (the 55-bar section, letter H to K) also serve as the origin and termination respectively of the 72-bar and 116-bar modules of the other diagonally-related system.

3. The number of proportionally-unrelated bars encompassed by each system as a result of the first diagonal relationship is the same--42 bars—and the number of bars not included in the largest module—17 bars—is part of the golden section series of that module.

4. Most interestingly, in his own formal analysis of this movement, Bartok is ambivalent as to the beginning points of the recapitulation and coda. In the body of the analysis, he designates letter G to J as the recapitulation and letter J to the end as the coda, but in an

accompanying note he adds that "we could also designate _F_ as the beginning of the recapitulation, _I_ as the beginning of the coda."[14] Since _G_ and _J_ are the origin and termination of one of the golden sections on which a diagonal relationship is based, and _F_ and _I_, of the other, this would indicate that the elements of sonata form were not primary in the composer's intentions, but that a complex scheme of golden relationships was developed and a sonata form fitted loosely therein.

The existence of such a variety of proportional systems, all based on the principle of golden section yet going far beyond simple partitioning at the golden mean and the self-contained hierarchy of golden modules, indicates the operation of an intellectual curiosity searching for ever-new and more intricate applications of a basic formula and generating a wealth of formal interrelationships equal in richness to their musical content.

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[14] "Analysis for the Fifth String Quartet," Bela Bartok Essays, p. 414. See also Chapter VI, p. 139.

CHAPTER IV

A THEORY OF INSERTIONS AND OMISSIONS

The second movement of the Fourth Quartet presents some intriguing problems from which a revealing solution may be inferred.

Bartok's analysis of the formal structure of this quartet indicates that:

> The second movement has the character of a scherzo and is in three parts:
> measures 1 - 61: Part One;
> 62 - 77: transition;
> 78 - 188: Part Two (articulated as follows: 78 - 101, 102 - 136, 137 - 174, 175 - 188 as a retrogression up to Part Three);
> 189 - 222: Part Three (free recapitulation);
> 223 - 250: coda. 1

It seems clear that Part One, along with the transition to Part Two, is laid out in golden proportion: the first 15-bar section is divided into modules of nine and six bars; the next 62 bars are sectioned positively at the return of the Scherzo theme (in bar 54) into modules of 38 and 24 bars (Figure 69).

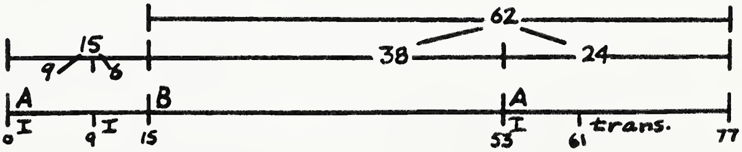

Figure 69. Golden proportion in Quartet IV, movement 2, Part One

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1 "Structure of the Fourth String Quartet," Bela Bartok Essays, p. 412. Walker's analysis subdivides the first part into an A-B-A with the dividing points after bars 15 and 53; he further indicates that the Scherzo theme is presented twice in the first A section--once in the first nine bars and again in the next six.

Part Three as outlined by Bartok is also 62 bars in length; if the coda begins in bar 223, it is divided in sections of 34 and 28 bars. However, Walker and Antokoletz see the coda as beginning in bar 213, at the point of negative golden section of this part, making it a proportional reflection of bars 16 to 77 of Part One (Figure 70).

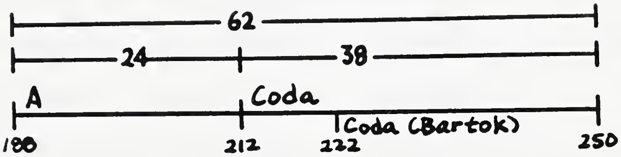


Figure 70. Golden proportion in Quartet IV, movement 2, Part Three

Bartok's articulation of Part Two yields sections of 24, 35, 38, and 14 bars in length. It seems significant that the numbers 24 and 38 are again in evidence (Figure 71).

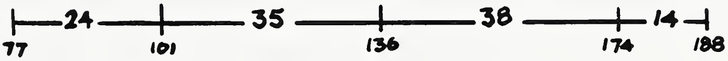


Figure 71. Subdivision of Quartet IV, movement 2, Part Two

The problem arises in trying to relate the proportions outlined above to a higher-level partitioning which would encompass the whole movement. That such a proportional system might exist is suggested by the use of the same golden series—62/38/24—in the proportioning of both Parts One and Three and in determining the lengths of two of the subsections of Part Two.

However, such larger-scale golden proportions as may be found seem more coincidental than logical. The whole of the movement from bar 10 onwards would divide in golden proportion after bar 101, the beginning of the second subsection of Part Two; however, this point and section do not appear to be of overriding significance to the whole movement, and the omission of the first nine bars was only arbitrary.

The first 188 bars divide by golden section after bar 72, the point which Thomason designates as the start of the second major section of this movement; however, this would contradict Bartok's assertion that this passage is transitional to bar 78, and would nullify the 62/38/24 proportioning of the first section which seems so obviously to belong to some larger system.

The presence of the 24- and 38-bar sections in Part Two and the apparently unrelated lengths of 35 and 14 bars of the other two sections suggest a possible explanation of this enigma. It is conceivable that the 35- and 14-bar passages--the second and fourth subsections of Part Two--were not included in an earlier draft of this movement. In that case, Part Two would also have been 62 bars in length, negatively sectioned in golden proportion after bar 101. The whole movement would then have consisted of three proportionally symmetrical sections of 62 bars each, preceeded by a 15-bar exposition of the Scherzo theme, a total of 201 bars. The first two portions--77 bars when taken together--would have balanced in golden proportion (and offset in polarity) the last two sections--124 bars together--the whole movement thus sectioned negatively at the beginning of Part Two (Figure 72).

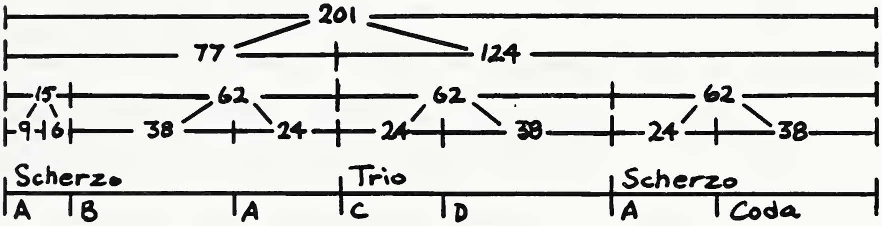
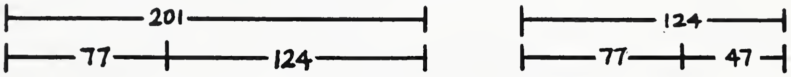


Figure 72. Possible early draft
of Quartet IV, movement 2

This leads to the further speculation that the initial intention was to relate the second movement to the fourth movement not only through the similarity of their forms (scherzo-plus-trio) and themes (rising and descending pattern) and by the converse relationship of their meters (6/8 and 3/4), keys (major third above and below the central key of C), and timbres (muted and pizzicato), but through the golden proportion of their lengths (201 bars divided as 77/124, and 124 bars divided as 77/47; Figure 73).



Movement 2 Movement 4

Figure 73. Possible golden proportion
of movement 2 to movement 4
in an early draft
of Quartet IV

So much more intriguing, then, is the enigma of the additional 35 measures in the middle and the 14 measures at the end of Part Two of the second movement. The 14-bar passage might be related in proportion to the 24-bar section which follows it, as it would be the next term in the golden series 62/38/24/14 on which the movement appears largely to have been based (Figure 74).

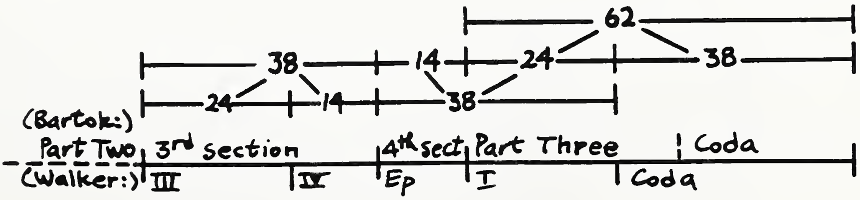


Figure 74. Golden systems in Quartet IV,
movement 2, bars 136-250

However, the 35-bar insertion is not so easily explained, as it does not relate in golden proportion to any other section of the movement. While several other hypotheses are possible,[2] the most probable explanation is, in accordance with Occam's razor,[3] the simplest: that in the course of composing the movement, Bartok felt that the second part was not of a weight appropriately to balance the other two, and added forty-nine bars to make it so. This would imply that the system of golden proportions laid out in advance was merely an aid to composition or an intellectual curiosity, and was abandoned when the craft of composition required it to be, since it was not intended for perception by the listener in the first place.

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[2] First, the additional forty-nine bars makes the second movement more nearly equal in chronometric length to the fourth. Second, and more esoteric, if the number of beats in each movement is determined, assuming that the 6/8 meter of the second movement is subdivided into two dotted-quarter-note beats and that no adjustment is made for the various tempos of the movements, the sum of the beats of the second and third movements is equal to the number of beats of the finale.

[3] Also called the Law of Parsimony: the logical principle that no more causes or forces should be assumed than are necessary to account for the facts; a corollary being that of several possible hypotheses, the one most likely to be true is that which requires the fewest assumptions, i.e., the simplest.

A similar case may be seen in the third movement of the Second Quartet. This movement has been variously analysed as "eight almost entirely independent sections laid end to end" (Gow[4]), a ternary structure (Walker and Thomason), and a sonata form (Lendvai). It has a total of 141 bars, and divides by positive golden section after 87 bars, at the beginning of the fifth section (Gow), fourth theme group (Walker), or recapitulation (Lendvai).

The printed score includes rehearsal numbers, placed an average of 12 or 13 bars apart, with one significant exception: rehearsal numbers 6 and 7 are 28 bars apart, which is twice the length of the next smaller span. Between them, but not demarcated by them, lies the 20-bar central section of the movement, bars 68 to 87 (Figure 75).

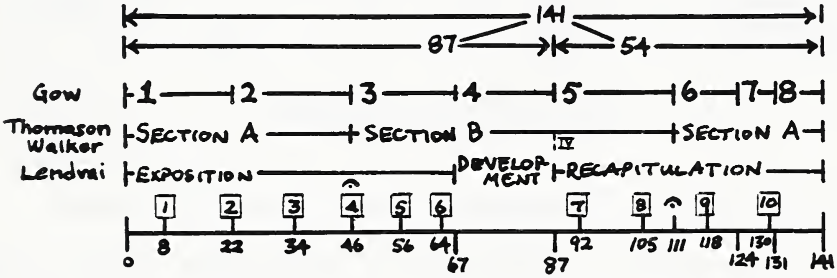
141
87
54
Gow
1
2
3
4
5
6
7
8
Thomason Walker
SECTION A
SECTION B
IV
SECTION A
Lendvai
EXPOSITION
DEVELOPMENT
RECAPITULATION
1
2
3
4
5
6
7
8
9
10
0
8
22
34
46
56
64
67
87
92
105
111
118
124
130
131
141

Figure 75. Sections and rehearsal numbers
in Quartet II, movement 3

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[4] David Gow, "Tonality and Structure in Bartok's First Two String Quartets," Music Review, XXXIV (1973), 269.

When graphed to scale, it appears visually obvious that the central section was inserted after the rehearsal numbers were placed. While this is not logically obvious—it would have been little trouble for the composer or the printer to reposition rehearsal numbers—it may be that this section is either intentionally or subconsciously differentiated from the others in this manner.

Significantly, if the 20-bar central section is omitted, the movement is reduced to 121 bars, and divides by negative golden section into modules of 46 and 75 bars at the beginning of the third section, which is marked by rehearsal number 4 and a fermata over the bar line. This is the section which both Walker and Thomason designate as the start of the B section of a ternary design; Fladt terms it a "funeral march," which returns in bars 125-131 (the seventh section; Figure 76).

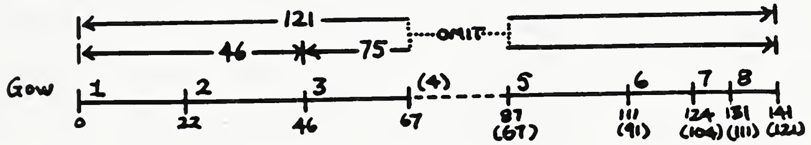

Figure 76. Omission theory
in Quartet II, movement 3

Again, it is possible that the original draft of this movement called for 121 bars divided in seven sections; at some point, an eighth section was added in the middle, of an appropriate length to shift the point of golden section to the end of the added portion.

A third example of an inserted passage is identified on the basis of a third type of internal evidence. In bars 328 to 331 of the first movement of the Sixth Quartet, the recapitulation of the second theme seems destined for a cadence: the lower voices decrease in rhythmic activity, arriving on a sustained E♭-B♭ chord; the violins descend from a high G in a variation of the second theme presented in octaves with rhythmic augmentation of the last notes, arriving on a full-bar trill on D; the B♭ descends to A♭ in the lower voices, seeking resolution on G; the upper voices complete the trill with a termination figure; these last bars are *un poco allargando* from the tempo *Meno vivo*, ♩. = ca. 120.

As the bar line to 332 is crossed, the tempo suddenly increases to *Vivacissimo*, ♩. = ca. 160; agitation replaces repose. Instead of resolving, each voice leaps to its next note--by a perfect fourth, a major seventh, a doubly-augmented fourth, and an augmented octave. The dynamic is *piano subito*. The passage so abruptly arrived at is the return of the closing group. The theme returns in a series of halting gestures, each arriving at a sustained triad--B♭ major in bar 343; B minor in 348; and D major in 352. After the theme is presented in imitation after bar 355, the section appears to be arriving at another D major triad, as the First violin sustains high D, followed by the second violin on A and the viola on F♯. The cello is about to arrive at D, but another sudden shift occurs.

Bar 363 brings an arrival at the key of the subdominant, G. The cello drops two-and-one-half octaves and the second violin, one-and-one-half octaves, to outline a G major triad. The spacing of this chord suggests that it is the resolution of the second theme group anticipated in bar 331, postponed by the intrusion of the closing group which demanded recapitulation. There follows a transition to the coda, which begins in bar 371.

This analysis of the recapitulation of the closing group as unprepared becomes the more fascinating when the proportional relationships of this section are examined. The recapitulation section from bar 268 to 370 (that is, including the "false reprise" but excluding the 20-bar coda) contains 103 bars, which divide by positive golden section (103/64/39) precisely at bar 332, the point at which the closing group abruptly breaks in.

In addition, if the 31-bar closing group is removed, a 72-bar section remains which again divides by positive golden section (72/44/28), now precisely at the beginning of the second theme group's recapitulation in bar 312 (Figure 77).

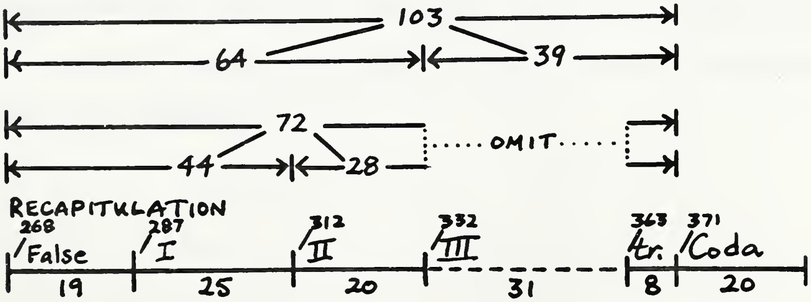

Figure 77. Omission theory in Quartet VI, movement 1

This case differs from the previous two in that the "removable" section does not appear to be an insertion due to a revision of an earlier proportional scheme, for it affects only the recapitulation and not the outline of the whole movement. Rather, the proportional scheme seems designed to reinforce the relationship of the various musical materials within it, as it emphasizes the sudden intrusion of the closing group in contrast to the more seamless transition between the first and second theme groups.

Thus while the two former examples seem to suggest that golden proportion was used as part of a working scheme in which the relative length of each section was laid out in advance but with the flexibility demanded by the musical material as it filled out the form, the third instance suggests that golden proportional designs were used either out of intellectual curiosity--the challenge of designing a mathematically-derived set of interlocking proportional systems, a source of satisfaction to the composer not necessarily meant to be shared with the perceiver--or due to a concern for musical architecture--a conviction that proportional relationships which correspond to the musical relationships they enclose would unify and vivify the musical structure, with each part felt to be of appropriate length and placement, whether or not the design was consciously perceived.

CHAPTER V

COINCIDENCE, INTUITION, AND THE FIBONACCI SERIES

Golden proportion can originate in one of four ways: First, it can be produced naturally, as by organic or geometric necessity; any organism which develops according to a logarithmic spiral (often called the "growth spiral") or any object with a structure based on a regular pentagon will naturally exhibit golden proportion.

Second, it can occur coincidentally. While the possibility is rather remote that a random subdivision will yield a part which is exactly 0.618 of the whole quantity, in other contexts the probability of coincidental golden section is quite high. In many instances, the probability that subdivision will occur at any particular point is not necessarily equal for all points of the quantity. In some cases, the probability might be highest for division at the center point and decreasing toward the extremes (Figure 78).

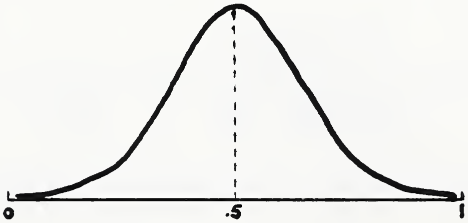

Figure 78. Center-weighted probability of division at any point

In other instances, the chance of division might be higher at a point (or points) other than the center, as though a "critical mass" or certain degree of imbalance must be reached (Figure 79).

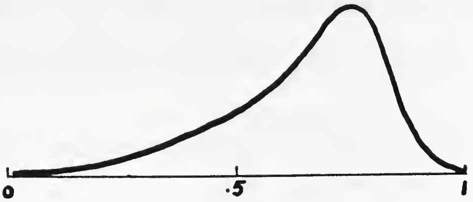

Figure 79. Other-than-center-weighted probability of division at any point

Furthermore, if the section occurs or is reported in discrete terms (i.e., in whole objects or numbers, rather than at an infinitesimal point in a continuum), the possibilities are limited and may be stated exactly. Thus, if a case is confined to a division yielding more than one-half but less than three-quarters, and is reported in whole-number percentages, the chance of the section being at the golden mean (62%) is one in twenty-four; if limited to more than one-half but less than two-thirds, the chance rises to one in sixteen[1] (Figure 80).

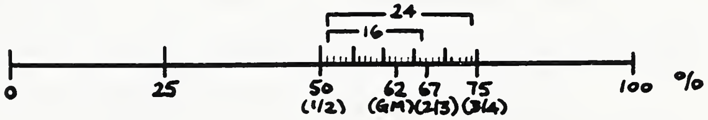

Figure 80. Probability of golden section (percentage points in a limited range)

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[1] In each case, equiprobable division within the specified range is assumed. The golden mean (0.618) is very nearly 5/8 (0.625), which is the half-way point between 1/2 and 3/4; thus, if the probability of division in this range is center-weighted, the chance of division at the golden mean is considerably higher.

Moreover, if the golden mean is seen to have a ±2% "area of influence" or "degree of error"--in effect, a range of five percentage points--the chance of section occurring in that range increases to one in five (between one-half and three-quarters) or better than one in four (between one-half and two-thirds; Figure 81).

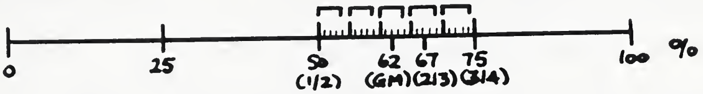

Figure 81. Probability of golden section
(percentage ranges)

The probability of golden section occurring coincidentally is similarly high when dealing with small whole numbers (e.g., cards in a suit; bars in a phrase group). The probability of eight dividing according to the golden (Fibonacci) series as five plus three is one-quarter, as there are only four possible divisions: four plus four, five plus three, six plus two, and seven plus one. Similarly, there is a one in five chance of eleven dividing as seven plus four, and a one in six chance of thirteen dividing as eight plus five. However, the larger the term, the lower the probability of its coincidental division according to a golden series; for instance, the chance of 377 being sectioned as 233 plus 144 is one in 188 (assuming all possibilities are of equal probability).

The third way in which golden proportion can originate is by intuition, the sense that a human creator might have that one pattern is more satisfying than another. Most of the psychological and aesthetic testing which has been done regarding the golden mean has studied its intuitive appeal.[2]

Similarly, most of the claims that golden section has been found to apply in music have assumed an intuitive origin. In his dissertation "Golden Proportion in Musical Design," Clive Pascoe hypothesized that "events of high musical significance occur more frequently on or near points of Golden Proportion than do events of low musical significance."[3] His analysis of fifty randomly-selected compositions revealed that 92% of them contained important events at the moment of golden proportion. In twelve of the sixteen sonata movements analyzed, the development began in the area of negative golden section; in eleven, the recapitulation began in the area of positive golden section.[4] Pascoe concluded that this "suggests a commonality among composers transcending period or stylistic considerations"[5] and that the optimum pattern of change and fulfillment was intuitively sensed.

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[2] See the discussion of Aesthetics in Chapter I, especially pp. 26-27.

[3] (University of Cincinnati, 1973), p. 41.

[4] Pascoe's measurements were based on the performance time of each section, and assumed a ±3% "area of influence" about the points of golden section.

[5] Op. cit., p. 224.

Fourth, golden proportion can be created intentionally. Composers and artists of our century have had a greater interest in symmetry and proportional schemes (both in creating and avoiding them, as by aleatory techniques) than those of at least the previous century. and many of them have experimented with or adopted the principle of golden section.[6]

In general, there are two characteristics which might identify those works in which golden section has been used intentionally as opposed to intuitively or coincidentally. First, the principle would probably be used systematically: it would apply to more than one set of proportions, govern the whole (or most) of the work rather than one portion, and be used at various levels, creating a hierarchy of proportional relationships. Second, the Fibonacci (or, more rarely, Lucas) series might be used to provide a whole-number approximation of a golden series which is relatively accurate to the smallest terms.

Of course, care must be exercised in the analysis of such a work to be sure that the system discovered actually has as its basis the principle of golden section and not some other principle, that the golden series identified is relevant in the original context of the work, and that the terms of the series used are large enough reasonably to exclude their coincidental occurrence.

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[6] See Chapter I, especially pp. 23-24, 29-30; see also David Alderton Caudill, "Sectional Proportionality as an Historical Parameter in Music" (Ph.D. dissertation, Florida State University, 1972).

Psalm 140[7] has thirteen verses. It is divided into stanzas separated by pauses (marked "Selah") after verses 3, 5, and 8, and, without a pause, after verse 11. Thus, the entire psalm is subdivided according to golden section, using numbers belonging to the Fibonacci series (Figure 82).

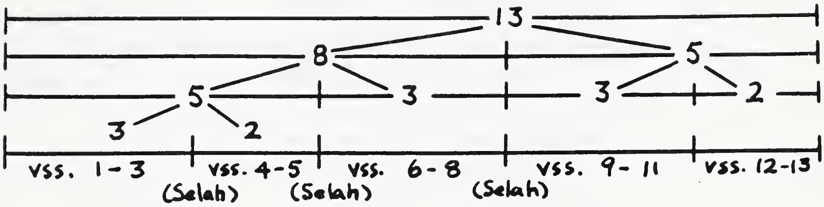

Figure 82. Golden section and
the Fibonacci series
in Psalm 140

However, this is probably entirely coincidental, as the numbering of the verses and their composition were entirely separate acts.[8] Furthermore, as the translation in the Jerusalem Bible[9] suggests, verses 4 and 5 may be inappropriately numbered, being really three verses like the other stanzas. In that case, the psalm consists of four stanzas of three verses each, with a concluding stanza of two verses, a total of fourteen verses (Figure 83).

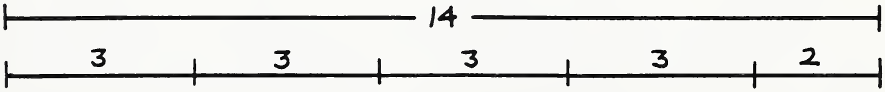

Figure 83. Probable original division
of Psalm 140

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[7] Vulgate: Psalm CXXXIX

[8] Verse numbers were first provided by St. Jerome in his Latin translation.

[9] Reader's Edition (Garden City, N.Y.: Doubleday, 1968), p. 802.

Similarly, there is no justification for analyzing a Baroque dance pair or a Classical Minuet and Trio as being "almost golden" when the fundamental principle of their structure is clearly that of two contrasting sections of equal (or almost equal) length, the first being repeated at the end (Figure 84).

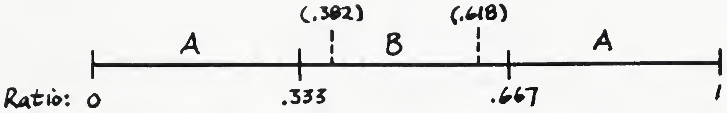

Figure 84. Ternary form compared with golden proportion

Even if a coda extends the final section, placing it in almost exact golden proportion to the first two sections, the claim that the composer intuitively sensed that the final section should begin at the golden mean may not be legitimate, as the sensation may merely have been that a more conclusive ending was required (Figure 85).

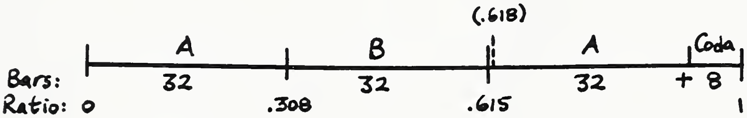

Figure 85. Ternary form with coda compared with golden proportion

Likewise, an eight-part form of equal-length sections (including a rondo in which the first section is repeated) should not be explained as an "almost exactly golden" structure, as it naturally divides at 5/8 or 0.625 of its length, only 0.007 away from the golden mean[10] (Figure 86).

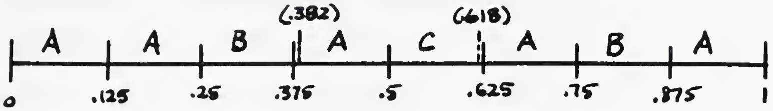

Figure 86. Eight-part form compared with golden proportion

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[10] In contrast, the eight-part form used by Bartok in the third movement of his Second Quartet divides after 87 bars of a total of 141 bars, or at 0.617 of its length, which is closer to the golden mean than to the 5/8 point, which would be after 88 bars.

The present study has sought to avoid identifying golden relationships which are probably coincidental, or forcing an interpretation based on golden section onto a movement based on other premises, by beginning with analyses of musical relationships, and by concentrating on systems of golden proportion which together encompass all or most of each movement. Nonetheless, it is on occasion difficult to discern which relationships are coincidental and which intentional, especially when the various analysts disagree on the form or dividing points of a particular movement. A case in point is the second movement of the First Quartet. While Walker and Thomason both see it as a sonata form with reversed recapitulation (although disagreeing on the start of the development section), Fladt sees it as having no development, but a developmentally-expanded reprise, whereas Gow sees it as an exposition and development with no recapitulation. The movement contains a veritable plethora of golden relationships, but few which coordinate with others and none which explain the whole movement.

In contrast, the first movement of the Sixth Quartet presents the clearest possible evidence of intentional golden section. The whole movement (excluding the Mesto theme which introduces each movement of the quartet) consists of 377 bars, the number 377 being the fourteenth term of the Fibonacci series. The beginning of the development divides the movement by negative golden section, with 144 bars in the exposition and 233 bars in the remainder of the movement, these being the twelfth and thirteenth terms of the Fibonacci series.

As though subtly confirming the use of the Fibonacci series, the development section is 110 bars in length, positively divided by the pitch climax in bar 226 into sections of 68 and 42 bars—110, 68, and 42 being double the Fibonacci terms 55, 34, and 21 respectively.

The codetta which ends the exposition (bars 137-57) is 21 bars in length, positively sectioned at bar 150 as 13 bars and 8 bars, thus using the sixth through eighth Fibonacci terms. The remainder of the exposition is thus 123 bars in length, the same length as the recapitulation (including the "false recapitulation" beginning in bar 268 and the coda; Figure 87).

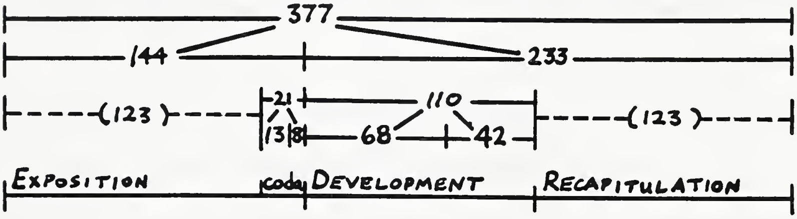

Figure 87. The Fibonacci series
in Quartet VI, movement 1

In addition to the obvious symmetry of beginning and end, this may be significant in that 123 is the tenth Lucas number—Bartok may have been using two of the many interrelationships between the Fibonacci and Lucas series[11] to complete the golden proportioning of the movement within the overall Fibonacci framework. (However, he may also have happened upon this scheme by coincidence, unaware of the significance of the Lucas series, as no other terms of the series are used in the movement.)

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[11] See Appendix II, equations 12 and 22, p. 171, 172.

Studies of the manuscripts and working drafts of the Sixth Quartet have yielded some valuable information about its composition.[12] First, they confirm that the introductory Mesto section was not conceived of as part of the first movement, but was added as a unifying and framing device at a later stage:

> What we can see from the sketch full score of the quartet is that Bartok began at the "Vivace" [bar 24] (though it is not so marked), and on completion of that, went on immediately to the "Marcia" [bar 17] (again not so marked) of the second movement, without introductory material. The third movement however is preceded in the sketch by its introduction, and it therefore seems likely that it was during the composition of the "Marcia" that Bartok either first thought of the prefatory subject, or first had the idea of using it to introduce each movement. 13

Second, they reveal two revisions of particular interest in the context of Fibonacci systems. The first is an eight-bar sketch of bars 39 to 46; "it was to be inserted in place of a five-bar passage that he had only half finished."[14] The second shows

> what appears to be a gradual working back of the first movement's introduction, from the end to the beginning. First Bartok wrote the octave passage that finally became bars 19-23, here marked 'Lento', and preceded by the Hungarian word 'eleje', meaning 'beginning'. Then he added a further five bars of octaves which in a slightly amended form became bars 14-18, with an arrow to show that these were to precede the first octave passage. 15

The point at which the first revision ends corresponds with the beginning of a contained Fibonacci system, bars 47-80; also, it is only the eleven-bar passage containing this revision which is not in golden proportion with some other section of the first theme group (Figure 88).

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[12] John Vinton, "New Light on Bartok's sixth Quartet," Music Review, XXV (1964), 224-38; Benjamin Suchoff, "Structure and Concept in Bartok's Sixth Quartet," Tempo, LXXXIII (Winter 1967-68), 2-11.

[13] Suchoff, p. 5.

[14] Vinton, p. 228.

[15] Suchoff, p. 5.

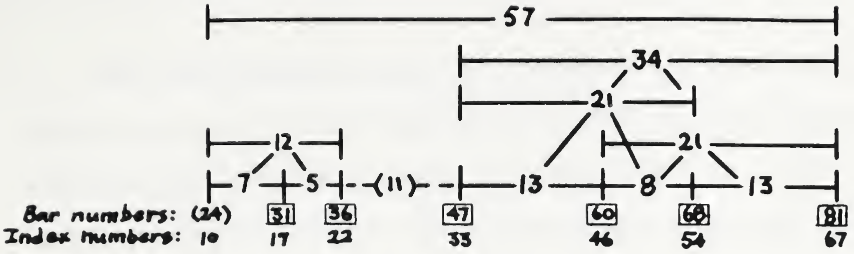
57
34
21
12
21
7
5
(11)
13
8
13
Bar numbers: (24) 31 36 47 60 68 81
Index numbers: 10 17 22 33 46 54 67

Figure 88. First theme group,
Quartet VI, movement 1

This expanded 57-bar section corresponds in golden proportion to the next thirty-six bars, containing the second theme group and the first part of the closing group. Neither the ten-bar introduction which was later added to the beginning of the movement nor the remainder of the exposition is related to any larger proportional scheme (although both the introduction and the final twenty-one bars of the exposition subdivide in golden proportion internally; Figure 89).

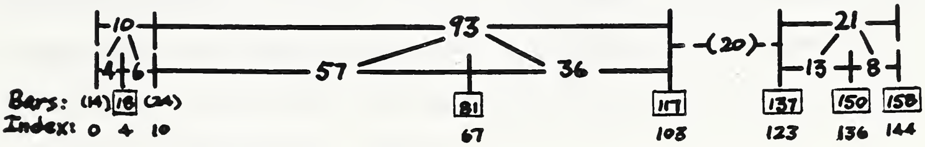
10
93
21
4
6
57
36
(20)
13
8
Bars: (14) 18 (24) 81 117 137 150 158
Index: 0 4 10 67 108 123 136 144

Figure 89. Exposition,
Quartet VI, movement 1

It might be inferred from the foregoing that Bartok began with a preconceived proportional scheme which was altered in the course of various musical revisions, resulting in an exposition which is in golden proportion to the rest of the work but is not rigorously proportioned within itself.

The other instance in which the Fibonacci series is used in an exposed way is the third movement of the Fourth Quartet, which was discussed as a linearly-extended system in Chapter III (p. 97). It appears less obviously in the fifth movement of the same quartet, in which the second section is 89 bars in length, sectioned negatively into 34- and 55-bar passages. A smaller but more extensively subdivided Fibonacci hierarchy is found in the return of the Burletta, bars 97 to 130 in the third movement of the Sixth Quartet: five modules divide the 34-bar passage (34/21/13/8/5/3).

It was also pointed out in Chapter III (p. 100) that one of the diagonally-related systems in the first movement of the Fifth Quartet is a Fibonacci system, based on the terms 55, 34, 21, and 13. It is no doubt also significant that the other diagonally-related system uses the numbers 72, 44½, 27½, and 17, which are one-half the Fibonacci terms 144, 89, 55, and 34. The number of bars by which the diagonally-related span overlaps the rest of the system in each case is 42, which is twice the Fibonacci term 21 (a logical consequence of equation 2, Appendix I, p. 164).

In addition, the Fibonacci series governs the proportioning of the first theme in the exposition (to letter A; 13/8/5); the Lucas series sections the second phase of the development (letter E to bar 103; 18/11/7); numbers one and one-half times the above Fibonacci terms appear in the second phase of the exposition (letter B to C; 19½/12/7½, cf. 13/8/5); and terms double those of the Lucas series proportion the whole exposition (to letter D; 58/36/22, cf. 29/18/11).

The Third Quartet consists not of movements but "parts"—*Prima parte*, *Seconda parte*, *Ricapitulazione della prima parte*, and *Coda*—and is further unified by its proportional scheme. The *Prima parte* is in negative golden proportion to the exposition of the *Seconda parte* (to number 23; 293/112/181); the remainder of the *Seconda parte* is in positive golden proportion to the *Ricapitulazione* and most of the *Coda* (to number 10, excluding the final statement of each theme and a codetta; 382/236/146). As a logical consequence of the negative section of the first portion and the positive section of the second portion, the two central subsections, spanning the entire *Seconda parte*, are in golden proportion to the sum of the two flanking subsections (Figure 90).

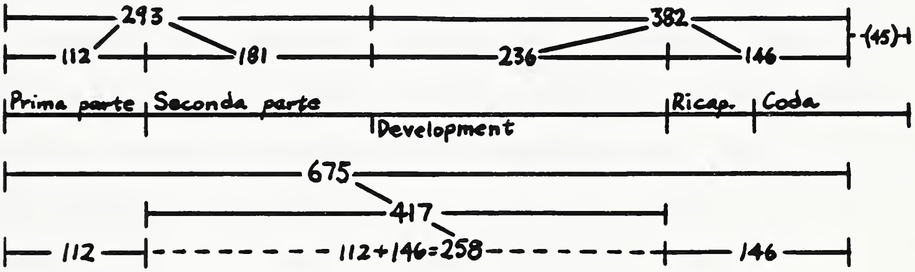
293
112
181
382
236
146
(45)
Prima parte
Seconda parte
Development
Ricap.
Coda
675
417
112
112+146=258
146

Figure 90. Golden section
of Quartet III

The role of the Fibonacci series in this scheme is either coincidental or ingenious. First, the 89-bar coda of the Seconda parte is in golden relationship to the first 55 bars of the Ricapitulazione (to number 6; Walker sees the concluding portion of the Ricapitulazione as a varied return of the episodic material from numbers 10 and 11 of the Prima parte; Figure 91).

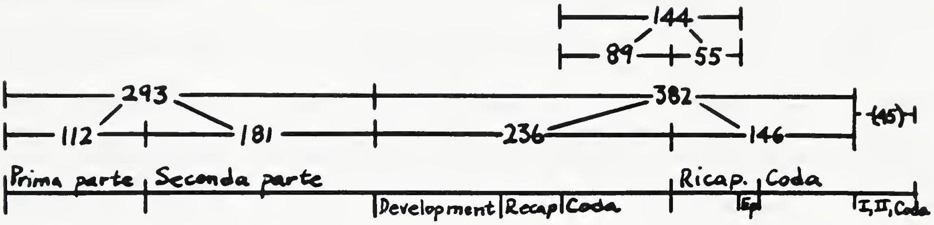
144
89
55
293
382
(45)
112
181
236
146
Prima parte
Seconda parte
Development
Recap
Coda
Ricap.
Coda
Ep
I, II, Coda

Figure 91. A Fibonacci system
in Quartet III

Secondly, the differences between the corresponding parts of the two major golden systems are Fibonacci numbers: the difference between the smaller sections is 34; between the larger sections, 55; and between the whole spans of each system, 89. Finally, the total number of bars in the quartet is 720, which is five times 144.

In summary,
146 - 112 = 34
236 - 181 = 55
382 - 293 = 89
720 ÷ 5 = 144

If this very subtle use of the Fibonacci series was intentional, it might have been part of a progressive willingness on Bartok's part to disclose his use of a principle of formal design: golden section. In the first three quartets, the application of this principle is disguised, rather than disclosed; although the evidence of its use would have been apparent to anyone who sought it, Bartok did not draw attention to his system either within the music or in his writing and speaking about it. In the third movement of the Fourth Quartet, however, the lower terms of the Fibonacci series are used prominently, with a sixteen-bar conclusion as though comparing the two proportional systems. Finally, in the first movement of the Sixth Quartet, its larger terms are used, effectively eliminating the possibility of their coincidental occurrence.

CHAPTER VI: CONCLUSIONS

The fundamental conclusion of this study is that while Bartok used the principle of golden section in the formal proportioning of his String Quartets, he did not often use it in the simple, modular hierarchy most analysts seek to find (Figure 92).

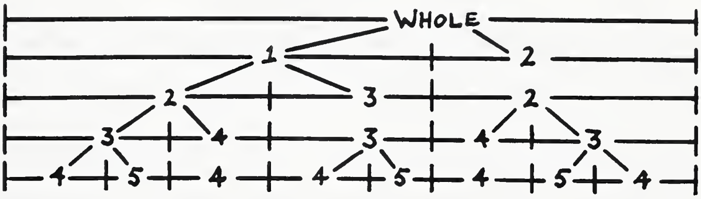

Figure 92. Modular hierarchy

Even when such a hierarchy is used, it usually is with subtle irregularity. For example, the Introduzione preceeding the third movement of the First Quartet is a modular hierarchy,[1] the whole span divided into two modules, each of those divided into two smaller modules, forming a symmetrical partitioning at the lowest level; but because the passage is thirty-three bars in length, rather than the Fibonaccian thirty-four which would neatly subdivide in a whole-number recursive approximation of a golden series, the central subsection is not a module of the same series as all the others, introducing an asymmetry at the lowest level which, by its central location, does not upset the symmetry of the whole (Figure 93).

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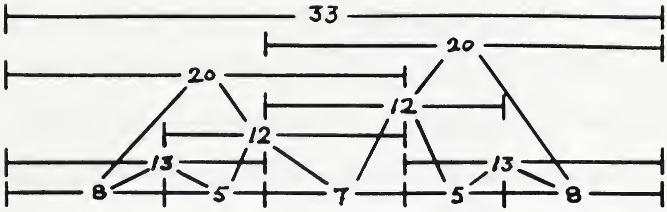
[1] See discussion in Chapter III, p. 91.

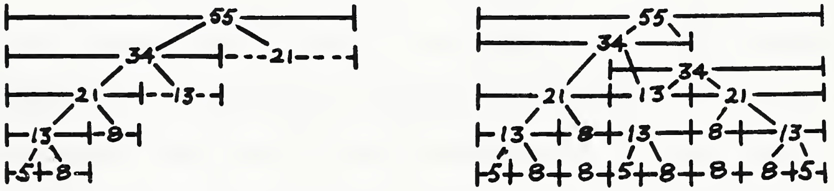


Golden series:
33/20.4/12.6/7.8/4.8
20/12.4/7.6/4.7
12/7.4/4.6

Whole-number approximation:
33/20/13/8/5
20/12/8/5
12/7/5
(non-recursive)

Figure 93. Modular hierarchy in Quartet I, Introduzione to movement 3

Similarly, the hierarchy of golden modules in the third movement of the Fourth Quartet was described in Chapter III (p. 97) as a linearly-extended system, since the smaller modules are presented first, combining to create larger modules, but the subsequent larger modules are not subdivided to complete a symmetrical set of smaller modules at the lowest level (Figure 94).



a. Linearly-extended system in Quartet IV, movement 3

b. Complete modular hierarchy

Figure 94. Comparison of linearly-extended system and modular hierarchy

Furthermore, the movement concludes with two eight-bar phrases, apparently unrelated to (or perhaps contrasted with) the proportional scheme governing the rest of the movement.[2]

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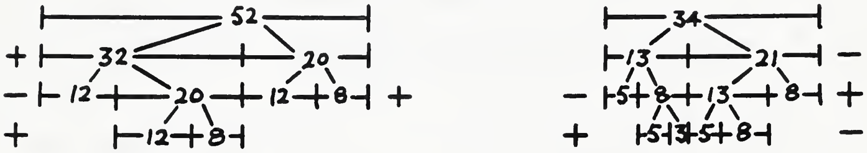
[2] In addition, there is an unexplained division after bar 41, and the change to Part Three actually occurs in the middle of bar 55, or after $54\frac{1}{2}$ bars.

The clearest uses of a modular hierarchy are in the first movement of the First Quartet and the third movement of the Sixth Quartet. Although each system governs only a portion of a movement (the first two sections of a ternary design in the former case and the return of the Burletta in the latter), they use five and six modules respectively at four levels. The latter example is a Fibonacci system, and is balanced in polarity at both lower levels; it is opposite in polarity to the former system (Figure 95).

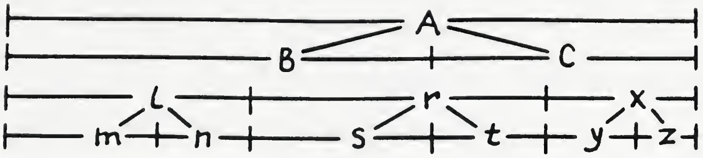


a. Modular hierarchy in Quartet I, movement 1

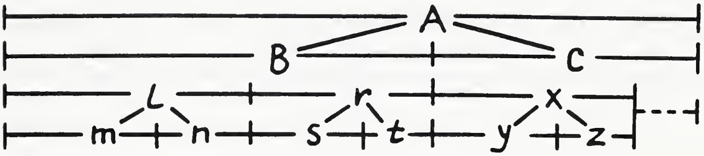
b. Fibonacci modular hierarchy in Quartet VI, movement 3

Figure 95. Modular hierarchies governing portions of movements

More characteristic of Bartok's approach is a two-level structure of aligned but proportionally unrelated golden systems: One set of golden modules proportions the entire movement, while three other sets govern each of the main sections of the movement. The four sets are usually proportionally unrelated to each other, but are aligned so that the point of golden section of the whole movement coincides with either the golden mean or a terminal point of one of the smaller sets (Figure 96).



a. Aligned at point of golden section of A and r



b. Aligned at terminal point of r

Figure 96. Alignment of proportionally unrelated golden systems

The former type is found in the first movements of the first two quartets and, in more elaborate form, in the third movement of the First Quartet. The latter type describes the forms of the last two movements of the Fourth Quartet, the third movement of the Fifth Quartet, and the first movement of the Sixth Quartet. Typically the three smaller systems do not cover exactly the same span as the larger system—omitting a coda, perhaps—since otherwise one of the smaller sections would produce modules of the same golden series as the largest system (Figure 74).

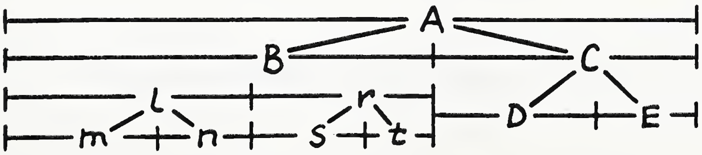


Figure 97. One smaller system as modules of the larger system

The form of the first movement of the Fourth Quartet is a variant of the two types described above, in which the largest system does not include the whole movement, and only two smaller systems are used at the second level (Figure 98). The golden mean and the terminal point of the largest system each align with the golden mean of one of the smaller systems. The Finale of the Fifth Quartet is simpler still, with only one smaller system, and does not include the first section of the movement in its proportional scheme.

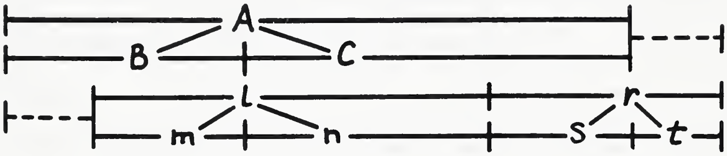
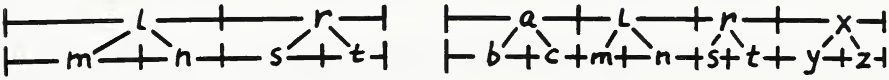


Figure 98. Variant of typical proportional schemes

The proportional schemes used in the Third Quartet, the first movement of the Fifth Quartet, and the middle movements of the Sixth Quartet are similar in their division of the movement into several smaller golden systems (two or four in these cases), but lack an overall system sectioning the whole movement[3] (Figure 99).



a. Two-system scheme

b. Four-system scheme

Figure 99. Schemes with no overall system

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[3] The two-system schemes do not govern the entire movements; in the four-system schemes, one of the systems is linearly-extended or diagonally-related, i.e., other than a simple golden section of a span.

A number of more complex golden systems were discussed—interlocking, overlapping, linearly-extended, and diagonally-related systems—but these tend to be used within the framework outlined above, replacing one of the simple golden systems. The fifth movement of the Fourth Quartet, the first movement of the Fifth Quartet, and the third movement of the Sixth Quartet are clear examples of such usage.

The second movement of the Fourth Quartet is the only one in which systems and modules of the same size are used in different sections of the movement. This led to the speculation (discussed in Chapter IV, pp. 107-109) that the final form of the movement was the result of two insertions to a preliminary scheme in which three equal systems with a brief introductory system formed a larger golden system governing the whole movement (Figure 100).

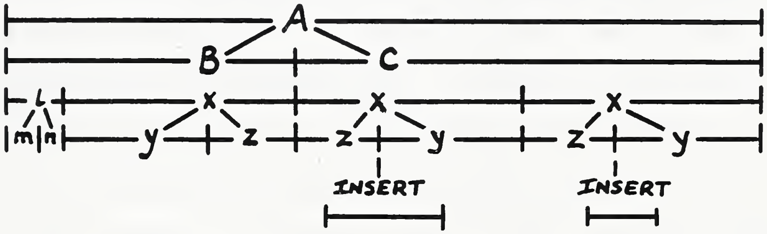

Figure 100. Insertions to a set of equal systems

Similarly, the third movement of the Second Quartet could have resulted from the insertion of a section in the middle of the movement; alternatively, it could have been designed as a structure displaying golden proportion both as is and with the middle section omitted (Figure 101).

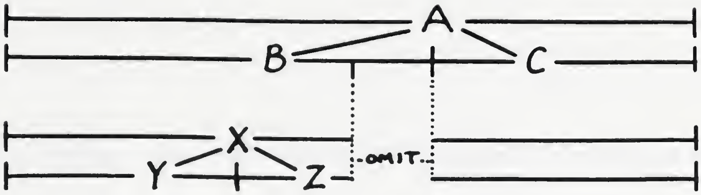

Figure 101. Omission from a golden system resulting in another golden system

Bartok's description of this movement as "a somewhat expanded A - B - A form"[4] gains new significance in light of this theory that it is quite literally an expansion of an earlier draft which also displayed golden proportion.

The second movement of the Fifth Quartet was discussed as the only one to demonstrate proportioning (golden and otherwise) both by bars (and half-bars) and by beats. While the overall movement displayed golden section in both systems of measurement, at the lower level a system based on beats sectioned the first part of the movement at a musical climax (the turning point in volume and the first use of the theme in inversion) and an interlocking system measured in bars marked the dividing points between the various sections of the larger part of the movement. In addition, the two major sections of the movement (A and B) were found to contain an equal number of beats, while the last section (A') was exactly half as many beats in length.

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[4] Cited in Szabolcsi, Introductory study to Bela Bartok: His Life in Pictures, p. 64.

This finding--that Bartok characteristically divided a movement into a number of sections, each in golden proportion within itself but proportionally unrelated to each other, with the whole movement often governed by a larger golden system aligned at some point with one of the smaller systems--may reinforce Anthony Payne's description of Bartok's music as "the static block-arrangement of sections":

> To Bartok, the sonata is not dynamic, though in places it does give that appearance; it is a matrix on which to work out an almost pathological concern with symmetries in architectural shape. . . . no type of motion is ever established as a norm. The disruptive tensions, in a free dissonance which often stems from persistent and closely interlocking symmetries, cancel each other out. Ostinati, if not actually present, are somehow implied by the hypnotically dissonant textures, and root the music to the spot. Consequently, sections which really do move are seen as independent blocks. There is movement within the block, but no flow beyond its limits into the next section, and it is fascinating to see how this concept of block architecture is adapted to sonata form. 1

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1 "The static and dynamic in Bartok," The Listener (April 2, 1964), p. 570.

This may also explain Bartok's apparent indifference toward the precise analysis of the sonata form in the first movement of the Fifth Quartet:

Movement I.: sonata form

1 – A + A – B	B – 36 + 37 – C	C – D
1st theme + appendix to 1st theme	transitional section + allusion to 1st theme	2nd theme
(in *B flat*)	(in *C*)	(in *D*)

exposition

D – F = development; leading to F – G (1st theme in inversion).
(in *E*)

G – J = recapitulation, in reversed order and in inversion, composed of:

G – H	H – I	I – J
(2nd theme in inversion)	(transitional section in inversion)	(1st theme in inversion)
(in *F sharp*)	(in *A flat*)	(in *B flat*)

J to the end: coda.

N.B.—(1) The sequence of tonalities of the single sections produce the whole-tone scale. (2) The first theme has two principal degrees: *B flat* (tonic) and *E* (dominant-like); the beginning, middle part, and end of the movement produce the following tonalities: *B flat, E,* and *B flat.* We could also designate F as the beginning of the recapitulation, I as the beginning of the coda. 2

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2 Bela Bartok Essays, p. 414; first published as "Analyse zum V. Streichquartett" in Muzsika, XIV/12 (December 1971), 26-28.

Bartok's indication that the recapitulation begins at either *F* or *G*, the coda at either *I* or *J*, suggests that a form other than the classic sonata form, albeit a similar or derivative form, was the basis of the composition. This is corroborated by the tonal scheme, which is based on the whole-tone scale (B♭, C, D, E, F♯, A♭, B♭), and the inversion and reversed order (in the primary analysis) of the themes on their recapitulation. It is clear that Bartok's primary concern was with "symmetries in architectural shape" rather than the opposition of two tonal areas resolved through recapitulation in only the tonic key; the heart of the concept is not conflict and development but "block architecture . . . adapted to sonata form."[3]

The use of golden proportion in this movement reflects this concern, corresponds to this concept: one golden system sections the exposition at the "allusion to the first theme"; two smaller systems govern *I* - *A* and *B* - *C* at a lower level. The development is proportioned in two sections (*D* - *E* and *E* - *F*), with one smaller system at the lower level. Both possible analyses as to the location of the recapitulation and coda are confirmed by the proportional scheme, one golden system covering *F* - *I*, another, *G* - *J*; both these systems are diagonally-related to larger modules which together partition the rest of the movement.[4]

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[3] Payne, op. cit., p. 570.
[4] See discussion of this movement in Chapter III, p. 100-103.

This view of Bartok's music as "the static block-arrangement of sections" goes far toward explaining why Bartok applied the principle of golden section as he did, rather than in large modular hierarchies which relate and proportion each part to the others, the whole system leading to and reinforcing one point of climax (Figure 92 above). Rather, each section is proportioned independently, for that is the way it operates musically; if a golden system spans the whole movement, it relates the lower-level systems to each other additively but not proportionately—the sum of two sections (or one section and part of another) forms a module of the upper-level system, but the sections are not in golden proportion to one another (and thus modules of the larger system; Figure 102).

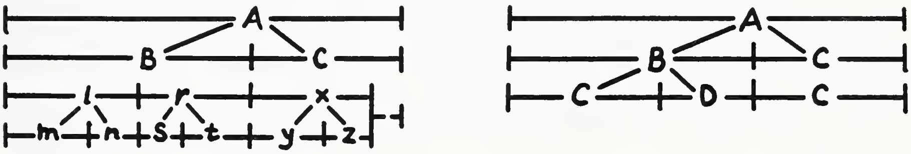

Figure 102. Bartok's system vs. modular hierarchy

The finding that the second movement of the Fifth Quartet includes separate but aligned systems of golden proportion, one measured in bars and one in beats, and that some of the proportions as measured in beats are not golden but rather in the ratios 1:1 and 2:1 raises a question which deserves further study: to what extent did Bartok integrate golden proportion with other systems of symmetry and proportion?

Another example of the use of a dual pattern of symmetry and asymmetry is found in the development of the first movement of the Sixth Quartet. The section in bars 194-230 features a series of graduated melodic peaks arrived at through an extended version of the first theme. The first important melodic peak, the first violin's high D in bar 213, is approached by a rising line of over two octaves in range, sustained for the full bar, and followed by a line descending over an octave. The next peak, on high E in bar 222, and the pitch climax of the whole movement, the high G in bar 226, are not approached from below but are followed by similar descending lines. The first peak occurs exactly at the geometric center or mid-point of the 110-bar development (bars 158-267), while the pitch climax occurs at the golden mean of the same span (Figure 103). Thus bilateral symmetry and dynamic symmetry are allied in the positioning of successive melodic peaks.

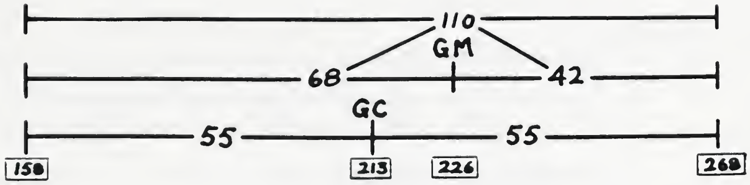

Figure 103. Geometric center and golden mean in Quartet VI, movement 1, development

A related symmetrical system may be seen in the fourth movement of the Sixth Quartet. From a piano beginning, it climbs to a melodic and dynamic climax after bar 12, then begins again, climbing to the end of bar 21. The next twelve bars climb further, reaching a sustained melodic climax on C# after bar 33. The twelve bars from here to the end of bar 45 conclude the Mesto introduction with a Piu andante which dies away ppp. As John Vinton has pointed out,[5] the next section is a return of bars 194-230 from the development of the first movement, the section based on graduated melodic peaks. In the last movement, it climbs to the end of bar 54, then climbs again, finally, after bar 66, reaching the same high G as was reached in the first movement, now emphasized through doublings at two lower octaves. The movement closes in the next twenty bars.

The resulting symmetrical structure consists of two groupings of thirty-three bars, each divided positively and negatively (but not quite by golden section) into parts of twelve and twenty-one bars (Figure 104).

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[5] "New Light," p. 228.

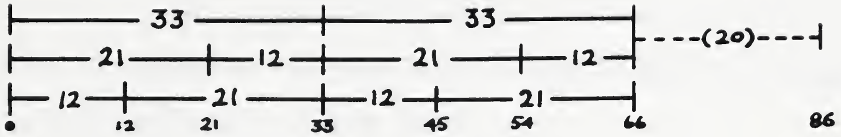
33
33
21
12
21
12
---(20)---
12
21
12
21
0
12
21
33
45
54
66
86

Figure 104. Symmetry and asymmetry
in Quartet VI, movement 4

Part of the fascination of this system is that 33 is seen to be the sum of two numbers, 12 and 21, which consist of the same digits transposed. In addition, 21 may have had numerological significance (being thrice seven) for Bartok, as it did for Schoenberg.[6]

This system integrates with golden proportion through the golden relationship of the final twenty bars with the previous twelve, these being golden modules of thirty-two. In addition, the nine-bar difference between the lengths of twenty-one and twelve, bars 46 to 54, is related in a contained system to the six-bar *Piu andante* and the final nine bars of the *Mesto* as golden modules of fifteen and twenty-four (Figure 105).

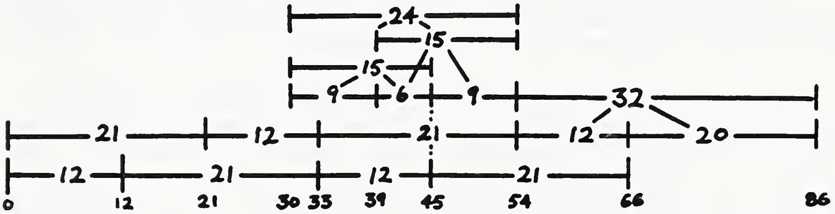
24
15
15
9
6
9
32
21
12
21
12
20
12
21
12
21
0
12
21
30
33
39
45
54
66
86

Figure 105. Integration of systems
in Quartet VI, movement 4

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[6] See Joan Peyser, *The New Music* (New York: Dell, 1971), pp. 3-12, 32.

This is similar to a finding outlined in Howat's criticism of Lendvai's analysis of the fugal first movement of Music for Strings, Percussion and Celeste (Figure 106a):

> The principal point of Golden Section falls after 55 bars, corresponding exactly with the dynamic climax (fff) of the movement on E♭, the farthest point from A on which the movement begins and ends. Intermediate divisions by Fibonacci numbers after 34 and 68 bars refer to the removal and replacement respectively of the string mutes, and that after 21 to the end of the fugal exposition. But this is an oversimplification. There are only 88 bars, not 89, in the movement; Lendvai's footnote to the effect that the movement 'must be completed by a whole-bar rest, in accordance with the Bülow analyses of Beethoven' is a shaky piece of reasoning.
> In addition to this the B♭ timpani pedal point and removal of string mutes take place after 33, not 34 bars, and the important celeste entry after 77 bars goes uncharted by Lendvai. (To conform exactly to Fibonacci proportions this last event would have to be placed after 76 bars, thus forming a division of 21:13 bars between the climax and the end in [Figure 106a].) Thus there are three displacements of a bar each, not mentioned by Lendvai: 34 to 33, 76 to 77 and 89 to 88, so that the movement is actually as represented by [Figure 106b]. While these deviations are not drastic enough to cause serious disturbance to the pattern of Golden Sections, they form (all being multiples of eleven) a subsidiary structure of simple ratios of eleven bars running through the movement.* There can be distinguished, then, two strands of proportional structure, each of independent character, one consisting of Golden Ratios and the other of simple whole-number ratios of eleven. This gives rise logically to the idea of a counterpoint of structure, with two or more formal plans running simultaneously.
>
> *Any special significance of the number eleven is not clear; one possible reason could be that the blocks of eleven-bar groupings give a constant periodicity while avoiding the squareness of four- or eight-bar periods. John Rutter, in an Open University course in the summer of 1975, pointed out a similar eleven-bar series running through the first movement of Mozart's G minor symphony, K. 550. 7

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7 "Debussy, Ravel and Bartok," p. 286-87.

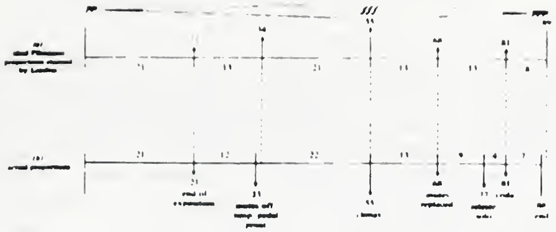

Figure 106. Proportions in Music for Strings, Percussion and Celesta

The most prominent use of multiples of eleven in the Quartets occurs in the fourth movement of the Fourth Quartet,[8] in which the golden section of the whole movement occurs after 77 bars, which is divided at the lower level into two parts of 44 and 33 bars. The manner in which this additive relationship between multiples of eleven is integrated with a contained system of golden proportions is truly ingenious (Figure 107).

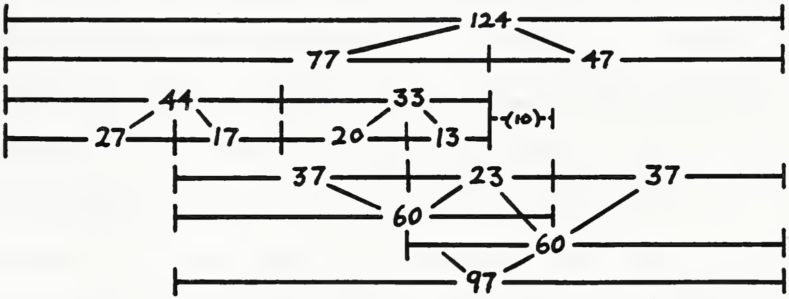

Figure 107. Multiples of eleven and golden section in Quartet IV, movement 4

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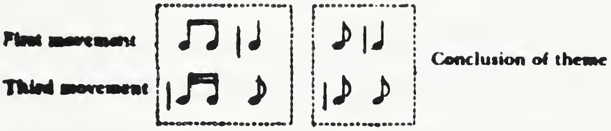
[8] In addition, the development section of the first movement of this quartet is 44 bars in length, the Scherzo of the second is 77 bars long, and the first two sections of the third movement total 55 bars, but these examples are isolated within each movement.

Ernő Lendvai has pointed to yet another mathematical relationship that may have had significance for Bartok:

> The circular course of the first movement of the Sonata for Two Pianos and Percussion Instruments is in no slight degree brought about by the "absolute" uneven measure (three times three eighths), while the third movement owes its static character to its "absolute" even measure (twice two eighths); in the second movement, even and uneven bars are intentionally made to alternate between the two. . . . In addition, the rhythms with a "strong" ending in the first movement have counterparts with "weak" endings in the third movement:



> Consequently, the themes of the first movement acquire a closed and those of the third movement an open form. The most interesting circumstance is that the dimensions of the complete work were not accidental—to quote only the final results, the time value of the composition (the above-mentioned 6,432 eighths = 804 whole notes) may also be traced to the symbol of the circle: $2^8\pi = 4^4\pi = 16^2\pi = 804$. 9

It may be that Bartok used the irrational number π (3.141593...) as consistently as the irrational ϕ (1.618034...). On the other hand, its appearance may be due to a coincidence of other factors: π is approximated by the rational number 22/7, which equals 11 x 2/7. Thus, if Bartok used the number 11 as was suggested above, it may be coincidental that even multiples of 11 will also approximate whole-number multiples of π; for example, the development section of the first movement of the Sixth Quartet is 110 bars in length, or $10 \times 11 = 5 \times 22 = 5 \times 7 \times (22/7) \doteq 35\pi$ bars.

---

9 "Duality and Synthesis," p. 113.

Of course, Bartok may intentionally have used this relationship between π and 11 for greater significance, be it numerological or purely mathematical. The third movement of the First Quartet may be 380 bars in length because $380 = 121\pi = 11^2\pi$. It may be significant that the basic unit length in the second movement of the Fifth Quartet is 41 beats, or 13π beats; moreover, the total length in beats is $223 = 71\pi$, divided by golden section as $138 = 44\pi$ and $85 = 27\pi$. Bartok may have found a relationship between π and the Fibonacci series in that $3 \times 5 \times 8\pi = F_4 \times F_5 \times F_6\pi = 377 = F_{14}$, and so chose to make the first movement of the Sixth Quartet 377 bars in length. The pitch climax in the fourth movement of the same quartet, which is also, as Vinton has pointed out, "the principal climactic point in the quartet,"[10] occurs after 66 bars, divided as two groups of 33, each doubly divided in sections of 12 and 21 bars. Since $66 = 3 \times 22 = 3 \times 7 \times (22/7)$, it closely approximates 21π. Lendvai's reference to a "circular course" in Bartok's music may be relevant here as well, as the climax not only occurs in a return to the cyclic theme which has preceded each movement, but on a return to the pitch climax of the first movement.

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[10] "New Light," p. 230.

This leaves the question first raised in Chapter II: Did Bartok expect his use of golden proportion to be perceived, assuming that the bar-length, and not the beat or the second of performance time, was the primary unit by which man perceives the passing of time in music? Did he use golden section as a compositional aid, a method of sketching out the form of a movement before infusing it with musical content? Or did Bartok design his intricate, interconnected systems of golden architecture merely for his own enjoyment, for the challenge of creating a mathematical substructure, or for the intellectual pleasure of knowing that he shared a principle of construction with the Greeks and even Nature herself?

The answer is no doubt a blend of all three possibilities: because of an interest in things mathematical and a concern that his music be based on a natural phenomenon, Bartok used golden section as an aid in the proportioning of each part and sometimes the whole of each movement, expecting that thereby the result would be perceived as balanced, architecturally sound, and satisfying.

It is clear that Bartok was not primarily concerned with the listener perceiving that the climax was placed at the perfect point, creating dynamic symmetry which would be instinctively appealing to the hearer. First of all, he typically positioned at the golden mean not the climax of a movement but the start of a new section. Secondly, he did not use smaller modules to reinforce in a golden hierarchy the principal point of golden section. Thirdly, he invented

complex systems of golden interrelationships which he could not have expected to be perceptible even to a listener aware of the use of golden proportion.

Rather, the results suggest that Bartok used golden proportion to impart logic and unity to the overall structure of each movement, and to reinforce musical relationships between and within formal sections. For example, he may have determined in advance of its composition that the exposition of the first movement of the Second Quartet would balance (in negative golden proportion) the remainder of the movement; that the statement phase of the first theme group would balance (positively) the initial development of the theme in the following bars; that the first theme group would not be in golden proportion to the following section nor to the whole exposition, but rather would form a self-contained block related only additively to the remainder of the exposition in the proportioning of the whole movement (Figures 32 and 96a above).

The theory of insertions advanced in Chapter IV suggests that if Bartok did sketch out movements in this manner, he did not feel bound to maintain his original proportional scheme if it conflicted with the musical events which came into being as the movement unfolded. This attitude toward golden proportion as an aid to composition and not a formula could explain the presence of many sections which seem proportionally unrelated to their surroundings, in addition to the specific examples discussed in support of the theory.

Perhaps the most intriguing example of the use of golden architecture to reinforce musical relationships is that of the unprepared recapitulation of the closing group in the first movement of the Sixth Quartet, a section which appears to be proportionally "removable"--the surrounding sections are in golden proportion both with and without the musically intruding section (pp. 112-114 above).

Surely such a double structure was not meant to be perceived by a listener, consciously or unconsciously, yet just as surely it is intentionally there. One is not meant to hear it, yet the structure ratifies the relationship.

One is meant to hear organized sound, not organization.

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III. Essays

Preceding the selected bibliography of essays relevant to the present topic is an alphabetical listing (according to abbreviation) of five publications containing Bartok essays. The abbreviations and their related pagination, essay number (BBE), or source number (BOM) are placed under the pertinent essay, along with references to other sources of the same essay. Titles in brackets indicate the original title, language and year of first publication (as given in BBE, Bibliography, pp. 527-555).

BBE Bartok, Bela. *Bela Bartok Essays*. Selected and edited by Benjamin Suchoff. (New York Bartok Archive Studies in Musicology, No. 8.) London: Faber and Faber, 1976.
A large collection of Bartok's writings, grouped by general topic; includes a bibliography of essays by Bartok in order of first publication.

BMR *Bela Bartok: A Memorial Review*. New York: Boosey and Hawkes, 1950.
A republication of the articles on and by Bartok featured in *Tempo*, Nos. 13 and 14.

BOM Vinton, John. "Bartok on his own Music," *Journal of the American Musicological Society*, XIX (1966), 232-43.
An article consisting of quotations from some of Bartok's writings, compiled from eight English sources, and placed into categories such as "Melodic Economy" and "Tonality and Modality."

BSE Reich, Willi, editor. *Bela Bartok: Eigene Schriften und Erinnerungen der Freunde*. Basel-Stuttgart: Benno Schwabe, 1958.
Includes five essays by Bartok.

BWW Szabolcsi, Bence, editor. *Bela Bartok: Weg und Werk; Schriften und Briefe*. Budapest: Corvina, 1957.
Includes seven essays and selected letters by Bartok; also, five essays on Bartok's life and work by Szabolcsi, Kodaly, and Lendvai, and a bibliography of compositions, books, studies, and articles compiled by A. Szollosy.

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BMR, 7-10.
BSE, 9-17.
BWW, 143-147.

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BBE, 85.
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*Denotes analysis graphed in Figures 28-50.

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Introduction to Bela Bartok: Masterpieces for the Piano. [MS drafted in 1945.]
    BBE, 61.
    BOM, 8.

Mechanical Music. [A gépzene, 1937.]
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The New Hungarian Art Music. [Unpublished lecture notes prepared in 1942-43.]
    BOM, 7.

On the Significance of Folk Music. [A népzene jelentőségéről, 1931.]
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    BMR, 74-76.
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*Denotes analysis graphed in Figures 28-50.

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APPENDIX I

FORMULAE RELATING TO SIMPLE RECURSIVE SERIES

Given any series of the form

(1) $$U_n = U_{n-1} + U_{n-2} ,$$

a number of relationships may be outlined.

First, since $U_{n+2} = U_{n+1} + U_n$

and $U_{n+1} = U_n + U_{n-1}$,

then $U_{n+2} = 2\ U_n + U_{n-1}$

(2) and $U_{n+2} - U_{n-1} = 2\ U_n$

or $\dfrac{U_{n+2} - U_{n-1}}{2} = U_n$.*

Similarly, since $U_{n+1} = U_n + U_{n-1}$

and $U_{n-1} = U_n - U_{n-2}$,

then $U_{n+1} = 2\ U_n - U_{n-2}$

(3) and $U_{n+1} + U_{n-2} = 2\ U_n$

or $\dfrac{U_{n+1} + U_{n-2}}{2} = U_n$.

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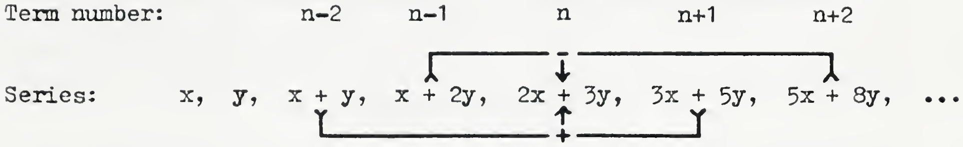
*As used in Quartet V, movement 1,

$F_{10} - F_7 = 2\ F_8$, or $55 - 13 = 2 \times 21 = 42$;

also, $\dfrac{F_{12} - F_9}{2} = F_{10}$, or $(144 - 34)/2 = 55$,

interlocking with the former equation to yield $(72 - 17) - 13 = 42$; see Figure 60 in Chapter III, p. 101.

These relationships may be illustrated as follows:



upper: $\frac{(5x + 8y) - (x + 2y)}{2} = 2x + 3y$

lower: $\frac{(3x + 5y) + (x + y)}{2} = 2x + 3y$

The sum of the first n terms of such a recursive series is given by:

$$U_1 + U_2 + U_3 + \dots + U_n = U_{n+2} - U_2 \,. \tag{4}$$

More generally, the sum of any sequence of consecutive terms is given by:

$$U_i + U_{i+1} + U_{i+2} + \dots + U_j = U_{j+2} - U_{i+1} \text{ for any } i, j; \tag{5}$$

that is, the sum of any number of consecutive terms plus the second such term will be another term of the series.

Further, the sum of the squares of the first n terms is given by:

$$U_1^2 + U_2^2 + U_3^2 + \dots + U_n^2 = U_n \cdot U_{n+1} + U_1^2 - U_1 \cdot U_2 \tag{6}$$

Similarly, the sum of the squares of any sequence of consecutive terms is given by:

$$U_i^2 + U_{i+1}^2 + U_{i+2}^2 + \dots + U_j^2 = U_j \cdot U_{j+1} + U_i^2 - U_i \cdot U_{i+1} \,. \tag{7}$$

Two numbers which have no divisors in common (except 1) are said to be "relatively prime." If the first two terms of any series of the form:

(1) $$U_n = U_{n-1} + U_{n-2}$$

are relatively prime, then any two consecutive terms of the series are relatively prime.

Proof by contradiction: Assume U_x and U_{x-1} have some common divisor $\underline{d} > 1$, but that U_1 and U_2 are relatively prime. Then $U_x - U_{x-1} = U_{x-2}$ is also divisible by $\underline{d}$. Likewise, $U_{x-1} - U_{x-2} = U_{x-3}$ must also be divisible by $\underline{d}$, and so on, until $U_4 - U_3 = U_2$ is divisible by $\underline{d}$, and $U_3 - U_2 = U_1$ is divisible by $\underline{d}$. But U_1 and U_2 are relatively prime, i.e., have no common divisor $\underline{d} > 1$. Having assumed the contrary of the assertion has led to a contradiction; therefore the assertion is proved.

Finally, any recursive series of this type approximates a series of golden means in its larger terms. In specific, the ratio between consecutive terms of any recursive series of the form:

(1) $$U_n = U_{n-1} + U_{n-2}$$

approaches $\emptyset$ as $\underline{n}$ approaches infinity.

Proof: Assume that $U_1 = p$ and $U_2 = q$. Then the series will be:

$$p, \quad q, \quad q + p, \quad 2q + p, \quad 3q + 2p, \quad 5q + 3p, \quad 8q + 5p,$$
$$13q + 8p, \quad \ldots, \quad F_{n-1}q + F_{n-2}p,$$

where F_{n-1} and F_{n-2} are terms of the Fibonacci series.

Thus the ratio between U_n and U_{n+1} will be:

$$\frac{U_{n+1}}{U_n} = \frac{F_n q + F_{n-1} p}{F_{n-1} q + F_{n-2} p} .$$

However, the ratio between F_n and F_{n+1} is:*

$$\frac{F_{n+1}}{F_n} = \frac{1/\sqrt{5}\left\{\phi^{n+1} - (-1/\phi)^{n+1}\right\}}{1/\sqrt{5}\left\{\phi^{n} - (-1/\phi)^{n}\right\}} = \phi - \frac{(-1)^{n+1}(\phi^2 + 1)}{\phi^{2n+1} - (-1)^n \phi} .$$

Since $\lim\limits_{n\to\infty} \frac{(-1)^{n+1}(\phi^2 + 1)}{\phi^{2n+1} - (-1)^n \phi} = 0$, then $\lim\limits_{n\to\infty} \frac{F_{n+1}}{F_n} = \phi$.

Thus, as n approaches infinity, F_n approaches $\phi \cdot F_{n-1}$, and F_{n-1} approaches $\phi \cdot F_{n-2}$, so that:

$$\lim_{n\to\infty} \frac{F_n q + F_{n-1} p}{F_{n-1} q + F_{n-2} p} = \frac{\phi \cdot F_{n-1} q + F_{n-1} p}{\phi \cdot F_{n-2} q + F_{n-2} p} = \frac{F_{n-1}}{F_{n-2}} = \phi . \qquad \text{Q.E.D.}$$

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*See Appendix II, equation 1a, p. 168.

APPENDIX II

FORMULAE RELATING TO THE FIBONACCI AND LUCAS SERIES

The analytical function for determining the value of any Fibonacci number (J. P. M. Binet, 1843) is:

$$(1)\qquad F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$

Since $\phi = (1+\sqrt{5})/2$ and $1/\phi = (\sqrt{5}-1)/2$, this may be restated as:

$$(1a)\qquad F_n = \frac{1}{\sqrt{5}}\left\{\phi^n - (-{}^1/_\phi)^n\right\}$$

Similarly, the formula for the n^{th} term of the Lucas series is:

$$(2)\qquad L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n = \phi^n + (-{}^1/_\phi)^n$$

Since the Lucas series approximates the ϕ series ($U_n = \phi^n$) and the Fibonacci series approximates the ϕ series divided by $\sqrt{5}$ ($U_n = \phi^n/\sqrt{5}$), the divergence of each term from its respective geometric series value is given by ${}^1/_{\phi^n}$ for the Lucas series and ${}^1/_{\sqrt{5}\phi^n}$ for the Fibonacci series; the direction of the divergence alternates between positive and negative. For both series, the ratio of this divergence to the value of the geometric series term is ${}^1/_{\phi^{2n}}$ (Table VI).

TABLE VI

Divergence of the Lucas and Fibonacci Series from their respective geometric series

Geometric Series	Recursive Series	Divergence	Ratio
Ø Series: $\varnothing^n$	Lucas Series: $\varnothing^n + (^{-1}/_{\varnothing})^n$	$\pm{}^{1}/_{\varnothing^n}$	$\pm{}^{1}/_{\varnothing^{2n}}$
Ø Series/√5: $\varnothing^n/\sqrt{5}$	Fibonacci Series: $\{\varnothing^n - (^{-1}/_{\varnothing})^n\}/\sqrt{5}$	$\pm{}^{1}/_{\sqrt{5}\varnothing^n}$	$\pm{}^{1}/_{\varnothing^{2n}}$

Since for the Ø series ($U_n = \varnothing^n$)

(3) $$U_n{}^2 = U_{2n} = \varnothing^{2n} ,$$

the divergence of the Lucas series from the Ø series may also be indicated by:

(4) $$L_n{}^2 = L_{2n} + 2\ (-1)^n ;$$

that is, the square of the $\underline{n}^{th}$ term of the Lucas series is within 2 of the value of the $\underline{2n}^{th}$ term. Since the Fibonacci series approximates the Ø series divided by √5, its similar statement of divergence is more complex:

(5) $$F_n{}^2 = \frac{F_{2n}}{\sqrt{5}} + \frac{2}{5}\left\{\left(\frac{-1}{\varnothing}\right)^{2n} - (-1)^n\right\}$$

(5a) $$= \frac{F_{2n}}{\sqrt{5}} - \frac{2}{\sqrt{5}}(^{-1}/_{\varnothing})^n\, F_n$$

The degree to which any three consecutive terms of the Fibonacci or Lucas series approximate a golden series may also be evaluated, as indicated by the following formulae:

(6) $$F_n^2 = F_{n+1} \cdot F_{n-1} - (-1)^n$$

(7) $$L_n^2 = L_{n+1} \cdot L_{n-1} + 5(-1)^n \;;$$

that is, the square of the mean is within 1 for the Fibonacci series or 5 for the Lucas series of the product of the extremes.

The relationship between any two corresponding terms of the Fibonacci and Lucas series (determined by comparing the formulae for their $\underline{n}^{th}$ terms as given in equations 1 and 2) is defined by:

(8) $$F_n = \frac{L_n}{\sqrt{5}} - \frac{2}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n = \frac{L_n}{\sqrt{5}} - \frac{2}{\sqrt{5}}\left(-1/\text{Ø}\right)^n$$

By comparison with equation 5, it is clear that the Lucas series (divided by $\sqrt{5}$) is related to the divergence of the Fibonacci series from the Ø series divided by $\sqrt{5}$. In fact, the following equation is derived by substitution:

(9) $$L_n \cdot F_n = F_{2n} \,,$$

indicating the complementary nature of the Fibonacci and Lucas series.

The more general equation expressing this relationship is:

(10) $$\text{for } n > p,\; L_n \cdot F_p = F_{n+p} + F_{n-p}$$

(10a) $$\text{for } n < p,\; L_n \cdot F_p = F_{n+p} - F_{p-n}$$

Interrelationships involving adjacent terms are also simple in nature, the following being among the most obvious:

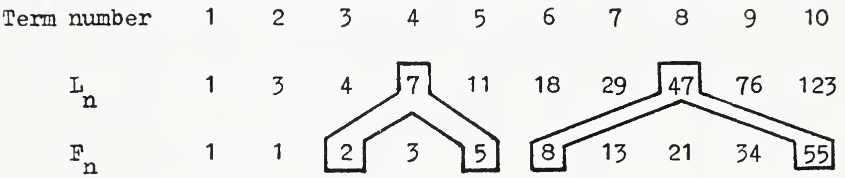
(11) $$F_{n+1} + F_{n-1} = L_n$$

(12) $$F_{n+2} - F_{n-2} = L_n{}^{*}$$

(13) $$L_{n+1} + L_{n-1} = 5\ F_n$$

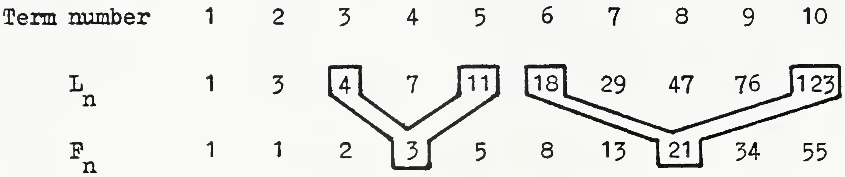
(14) $$L_{n+2} - L_{n-2} = 5\ F_n$$

as illustrated by the following:



Term number	1	2	3	4	5	6	7	8	9	10
L_n	1	3	4	$\boxed{7}$	11	18	29	$\boxed{47}$	76	123
F_n	1	1	$\boxed{2}$	3	$\boxed{5}$	$\boxed{8}$	13	21	34	$\boxed{55}$

(11) $5 + 2 = 7$ (12) $55 - 8 = 47$



Term number	1	2	3	4	5	6	7	8	9	10
L_n	1	3	$\boxed{4}$	7	$\boxed{11}$	$\boxed{18}$	29	47	76	$\boxed{123}$
F_n	1	1	2	$\boxed{3}$	5	8	13	$\boxed{21}$	34	55

(13) $11 + 4 = 5 \cdot 3$ (14) $123 - 18 = 5 \cdot 21$

Equations 11 to 14 are simple cases of a set of more general interrelationships:

For any n and m, $n > m$,

(15) $$\frac{F_{n+m} - (-1)^m F_{n-m}}{F_m} = L_n$$

(16) $$\frac{L_{n+m} - (-1)^m L_{n-m}}{5\ F_m} = F_n$$

(17) $$\frac{F_{n+m} + (-1)^m F_{n-m}}{L_m} = F_n$$

(18) $$\frac{L_{n+m} + (-1)^m L_{n-m}}{L_m} = L_n$$

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*As used in Quartet VI, movement 1: $F_{12} - F_8 = L_{10}$; $144 - 21 = 123$.

Given the simple case defined in equation 11 and the recursive definition of the Fibonacci series:

(19) $$F_{n+1} = F_n + F_{n-1} ,$$

the following corollaries are derived:

(20) $$L_n + F_n = 2\ F_{n+1}$$

(21) $$L_n - F_n = 2\ F_{n-1}$$

Taken one step further by a similar process, the following equations result:

(22) $$L_n + 2\ F_n = F_{n+3}\ ^{*}$$

(23) $$L_n - 2\ F_n = F_{n-3}$$

as illustrated by the following:

Term number	1	2	3	4	5	6	7	8	9	10
L_n	1	3	4	7	11	18	29	47	76	123
F_n	1	1	2	3	5	8	13	21	34	55

3+1=2•2 11-5=2•3 18+2•8=34 123-2•55=13

Given equations 9, 11, and 13, the following related equations are determined:

(24) $(F_{n+1})^2 - (F_{n-1})^2 = F_{2n}$ (25) $(L_{n+1})^2 - (L_{n-1})^2 = 5\ F_{2n}$

(26) $(F_{n+1})^2 + (F_n)^2 = F_{2n+1}$ (27) $(L_{n+1})^2 + (L_n)^2 = 5\ F_{2n+1}$

The more general equations expressing the same relationships are:

(28) $F_{n+1} \cdot F_{p+1} - F_{n-1} \cdot F_{p-1} = F_{n+p}$ (29) $L_{n+1} \cdot L_{p+1} - L_{n-1} \cdot L_{p-1} = 5\ F_{n+p}$

(30) $F_{n+1} \cdot F_{p+1} + F_n \cdot F_p = F_{n+p+1}$ (31) $L_{n+1} \cdot L_{p+1} + L_n \cdot L_p = 5\ F_{n+p+1}$

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*As used in Quartet VI, mvmt. 1: $L_{10} + 2\ F_{10} = F_{13}$; 123+2•55=233.

Inspection of the Fibonacci and Lucas series, the sum of the Fibonacci series, and the prime factorization of each term discloses several other interesting properties (Table VII).

First, every $\underline{n}^{th}$ term of the Fibonacci series is divisible by F_n; thus, for example, since $F_3 = 2$ and $F_7 = 13$, every third term is divisible by 2 (i.e., is even) and every seventh term is divisible by 13. The converse also appears to be true--no term has a Fibonacci factor unless its term number is a multiple of the term number of that factor; for example, 15 is a multiple of only 3 and 5, and $F_{15} = 610$ has as factors $F_3 = 2$, $F_5 = 5$, and 61 (which is not a Fibonacci number, in accordance with the principle stated).

Thus, while the general principle that consecutive terms are relatively prime* holds true for the Fibonacci series, a more specific principle can be enunciated: Any two terms of the Fibonacci series are relatively prime if their term numbers are relatively prime. (The converse--if two terms are relatively prime, then their term numbers are relatively prime--is not true, since $F_2 = 1$, and thus term numbers with only 2 as a common factor are also relatively prime; however, the principle that "two terms are relatively prime if and only if their term numbers have no common factors greater than 2" appears to be true.)

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*See Appendix I, p. 166.

A corollary of this principle is that a Fibonacci term with a prime term number is relatively prime to all other Fibonacci numbers except those with term numbers a multiple of the prime term number; for example, 19 is a prime number and $F_{19} = 4181$ —a product of 37 and 113—is relatively prime to all other Fibonacci numbers except F_{38}, F_{57}, F_{76}, F_{95}, and so forth. Conversely, if a Fibonacci number is prime, it will have a prime term number; for example, 89 is F_{11}, 233 is F_{13}, 1597 is F_{17}, and 28657 is F_{23}, all of these terms and term numbers being prime.

For the Lucas series, every 2n^th term following L_n is divisible by L_n; for example, $L_4 = 7$, and L_{12}, L_{20}, and L_{28} are divisible by 7. The converse would be that a Lucas number is divisible by a smaller Lucas number only if its term number is an odd-number multiple of the smaller term's number; that is, since 20 is five times 4, L_{20} is divisible by L_4, but since 16 is four times 4, L_{16} is not divisible by L_4 (but is, in fact, a prime number). This implies not only that a Lucas term with a prime term number will be relatively prime to all other Lucas numbers except those with term numbers an odd-number multiple of the prime term number, but that any Lucas term with a term number having only even factors (i.e., 2^n for any n) will be relatively prime to all other Lucas numbers; thus, L_8, L_{16}, and L_{32} have no factors in common with any other Lucas number (in fact, are prime numbers), since $8 = 2^3$, $16 = 2^4$, and $32 = 2^5$.

While the sum of the first n terms of the Fibonacci series is given by the general formula:

(32) $$F_1 + F_2 + F_3 + \dots + F_n = F_{n+2} - 1\ ,^*$$

another, more interesting relationship is made clear by the factorization of the series of sums--the sum of any number of terms of the Fibonacci series is a product of one Fibonacci number and one Lucas number, as given by the following equations:

(33) $$F_1 + F_2 + F_3 + F_4 + \dots + F_{4n-3} = F_{2n} \cdot L_{2n-1}$$

(34) $$F_1 + F_2 + F_3 + F_4 + \dots + F_{4n-3} + F_{4n-2} = F_{2n+1} \cdot L_{2n-1}$$

(35) $$F_1 + F_2 + F_3 + F_4 + \dots + F_{4n-3} + F_{4n-2} + F_{4n-1} = F_{2n} \cdot L_{2n+1}$$

(36) $$F_1 + F_2 + F_3 + F_4 + \dots + F_{4n-3} + F_{4n-2} + F_{4n-1} + F_{4n} = F_{2n} \cdot L_{2n+2}$$

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*Compare with Appendix I, equation 4, given that $F_2 = 1$.

TABLE VII

Fibonacci, Summation, and Lucas Series and Factors

TERM	FIBONACCI	PRIME FACTORS	SUMMATION	PRIME FACTORS	Fm·Ln	LUCAS
0	0	-	0	-	0	(2)
1	1	-	1	-	1·(1)	1
2	1	-	2	-	2·(1)	3
3	2	-	4	2^2	(1)·4	4
4	3	-	7	-	(1)·7	7
5	5	-	12	$2^2 \cdot 3$	3·(4)	11
6	8	2^3	20	$2^2 \cdot 5$	5·(4)	18
7	13	-	33	3·11	(3)·11	29
8	21	3·7	54	$2 \cdot 3^3$	(3)·18	47
9	34	2·17	88	$2^3 \cdot 11$	8·(11)	76
10	55	5·11	143	11·13	13·(11)	123
11	89	-	232	$2^3 \cdot 29$	(8)·29	199
12	144	$2^4 \cdot 3^2$	376	$2^3 \cdot 47$	(8)·47	322
13	233	-	609	3·7·29	21·(29)	521

TABLE VII—Continued

TERM	FIBONACCI	PRIME FACTORS	SUMMATION	PRIME FACTORS	Fm·Ln	LUCAS
14	377	13·29	986	2·17·29	34·(29)	843
15	610	2·5·61	1,596	$2^2\cdot3\cdot7\cdot19$	(21)·76	1,364
16	987	3·7·47	2,583	$3^2\cdot7\cdot41$	(21)·123	2,207
17	1,597	-	4,180	$2^2\cdot5\cdot11\cdot19$	55·(76)	3,571
18	2,584	$2^3\cdot17\cdot19$	6,764	$2^2\cdot19\cdot89$	89·(76)	5,778
19	4,181	37·113	10,945	5·11·199	(55)·199	9,349
20	6,765	3·5·11·41	17,710	2·5·7·11·23	(55)·322	15,127
21	10,946	2·13·421	28,656	$2^4\cdot3^2\cdot199$	144·(199)	24,476
22	17,711	89·199	46,367	199·233	233·(199)	39,603
23	28,657	-	75,024	$2^4\cdot3^2\cdot521$	(144)·521	64,079
24	46,368	$2^5\cdot3^2\cdot7\cdot23$	121,392	$2^4\cdot3^3\cdot281$	(144)·843	103,682
25	75,025	$5^2\cdot3001$	196,417	13·29·521	377·(521)	167,761
26	121,393	233·521	317,810	2·5·61·521	610·(521)	271,443
27	196,418	2·17·53·109	514,228	$2^2\cdot11\cdot13\cdot29\cdot31$	(377)·1364	439,204
28	317,811	3·13·29·281	832,039	13·29·2207	(377)·2207	710,647

All of these properties and interrelationships have enhanced the intrigue of the golden mean and the Fibonacci series. While the Lucas series provides the closest whole-number approximation of the Ø series, the Fibonacci series, approximating the Ø series divided by $\sqrt{5}$, is both the most basic recursive series and the most exact approximation of a series of golden means in the smallest terms: not only are its defining terms (the first two) equal to 1, it "originates" with zero--if a zeroeth term were to be determined for the Fibonacci series, it would be 0 (0 + 1 = 1, 1 + 1 = 2, ...) whereas it would be 2 for the Lucas series (2 + 1 = 3, 1 + 3 = 4, ...); the ratios between the smaller terms are remarkably close to Ø--the Fibonacci ratios 5/3 = 1.667 and 8/5 = 1.600 approximate 1.618 with only 3% and 1% deviation respectively (compared with the Lucas ratios 4/3 = 1.333 and 7/4 = 1.750, 18% and 8% deviation respectively).

These elegant mathematical and proportional qualities, combined with--and perhaps accounting for--its long history, its association with natural phenomena, and its wealth of applications in a variety of fields, have stimulated a wide interest in the Fibonacci series, and generated a lore surrounding it which is perhaps surpassed in the public mind only by that of the law of gravity (Newton and the apple) and the theory of relativity (Einstein's "$E = mc^2$").

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